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REGULAR ASYMPTOTIC METHODS IN CALCULATIONS OF THREE-LAYER PLATES†

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Two versions of the asymptotic method, which enables the initial problem to be broken down into two simpler problems which can be solved using the classical theory of plates, are considered for solving problems in the theory of three-layer plates in the formulation previously proposed (VOLOKH K. Yu., A theoretical-experimental investigation and optimization of three-layer plates supported at points. Candidate dissertation, Moscow, 1991). These methods are used for a static and dynamic analysis of a rectangular three-layer plate supported on hinges along the contour. Estimates are obtained for the errors of the approximate solutions obtained after an arbitrary number of iterations.

1. THE EQUATIONS of equilibrium in displacements and the natural boundary conditions of the theory of three-layer plates with a rigid filler that is uncompressed in a transverse direction were obtained by the author using the well-known broken-line hypothesis [1] and Lagrange's variational principle.

For a mainly bending deformation arranged symmetrically over the thickness of a three-layer plate for tangential displacements of the middle planes of the outer bearing layers ($i = 1, 2$) it is necessary to put $u_1 = -u_2 = u$ and $v_1 = -v_2 = v$; there are no tangential surface and external contour forces; $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$, $h_1 = h_2 = h$, where E_i are the moduli of elasticity, ν_i are Poisson's ratios and h_i are the thicknesses of the outer layers. In this case, the equations of equilibrium and the boundary conditions on the contour $x = \text{const}$ can be written in operator form as follows:

$$L_{11}(w) + L_{12}(u) + L_{13}(v) = q$$
$$L_{k1}(w) + L_{k2}(u) + L_{k3}(v) = 0, \quad k = 2,3 \tag{1.1}$$

$$L_1(w) + L_2(u, v) = 0 \leftrightarrow w = 0, \quad L_3(w) + L_4(u, v) = 0 \leftrightarrow \partial w/\partial x = 0$$
$$L_5(u, v) + L_6(w) = 0 \leftrightarrow u = 0, \quad L_7(u, v) + L_8(w) = 0 \leftrightarrow v = 0 \tag{1.2}$$

Here

$$L_{11}(w) = \bar{D}\nabla^4 w - Bc^2\nabla^2 w, \quad L_{12}(u) = 2Bc\,\frac{\partial u}{\partial x} - A_3\frac{h}{6}\nabla^2\frac{\partial u}{\partial x}$$

$$L_{22}(u) = 2A^*\frac{\partial^2 u}{\partial x^2} + A^*(1-\nu^*)\frac{\partial^2 u}{\partial y^2} - 4Bu, \quad L_{23}(v) = A^*(1+\nu^*)\frac{\partial^2 v}{\partial x\partial y}$$

$$L_1(w) = \bar{D}\left(\frac{\partial^3 w}{\partial x^3} + (2-\bar{\nu})\frac{\partial^3 w}{\partial x\partial y^2}\right) - Bc^2\frac{\partial w}{\partial x}, \quad L_2(u, v) = 2Bcu -$$

$$-A_3\frac{h}{6}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y} + (1-\nu_3)\frac{\partial^2 u}{\partial y^2}\right), \quad L_3(w) = \bar{D}\left(\frac{\partial^2 w}{\partial x^2} + \bar{\nu}\frac{\partial^2 w}{\partial y^2}\right)$$

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$$L_4(u, v) = -A_3 \frac{h}{6} \left(\frac{\partial u}{\partial x} + \nu_3 \frac{\partial v}{\partial y}\right), \quad L_5(u, v) = A^* \left(\frac{\partial u}{\partial x} + \nu^* \frac{\partial v}{\partial y}\right)$$

$$L_6(w) = -A_3 \frac{h}{12} \left(\frac{\partial^2 w}{\partial x^2} + \nu_3 \frac{\partial^2 w}{\partial y^2}\right), \qquad L_7(u, v) = A^* \frac{1 - \nu^*}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$L_8(w) = -A_3 \frac{h}{12}(1 - \nu_3) \frac{\partial^2 w}{\partial x \partial y}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$B = \frac{G_3}{h_3}, \quad A = \frac{Eh}{1 - \nu^2}, \quad D = \frac{Eh^3}{6(1 - \nu^2)}, \quad A_3 = \frac{E_3 h_3}{1 - \nu_3^2}, \quad A^* = A + \frac{A_3}{6}$$

$$\nu^* = \frac{\nu A + \nu_3 A_3/6}{A^*}, \quad \bar{D} = D + \frac{A_3 h^2}{12}, \quad \bar{\nu} = \frac{D\nu + A_3 \nu_3 h^2/12}{\bar{D}}$$

The operators L_{13} and L_{33} are obtained from L_{12} and L_{22} by replacing x and y and u and v; $L_{ij} = L_{ji}$; w is the flexure, E_3, ν_3 and h_3 are the modulus of elasticity, Poisson's ratio and the thickness of the middle layer (the filler) respectively, G_3 is the shear modulus of a transversally isotropic filler in planes perpendicular to the isotropy plane, and c is the distance between the middle planes of the bearing layers.

If we put $E_3 = 0$ in (1.1) and (1.2) we obtain the equations of three-layer plates with a light filler [2].

2. We will introduce a parameter ϵ into Eqs (1.1) and the boundary condition (1.2) such that $L_{12}^{(u)}$ is replaced by $\epsilon L_{12}(u)$, $L_{13}(v)$ by $\epsilon L_{13}(v)$, $L_2(u, v)$ by $\epsilon L_2(u, v)$, and $L_4(u, v)$ by $\epsilon L_4(u, v)$ and we will seek solutions in the form of series in ϵ

$$\mathbf{u} = \sum_{j=0}^{\infty} \epsilon^j \mathbf{u}^{(j)}, \quad \mathbf{u} = (u, v, w) \tag{2.1}$$

Substituting series (2.1) into the equations and boundary conditions and equating terms of like powers of ϵ, we obtain the iterative process

$$L_{11}(w^{(j)}) = -L_{12}(u^{(j-1)}) - L_{13}(v^{(j-1)}) + q^{(j)} \tag{2.2}$$

$$L_{k2}(u^{(j)}) + L_{k3}(v^{(j)}) = -L_{k1}(w^{(j)}), \quad k = 2, 3 \tag{2.3}$$

with boundary conditions when $x = \text{const}$

$$L_1(w^{(j)}) = -L_2(u^{(j-1)}, v^{(j-1)}) \leftrightarrow w^{(j)} = 0, \quad L_3(w^{(j)}) = -L_4(u^{(j-1)}, v^{(j-1)}) \leftrightarrow \leftrightarrow \partial w^{(j)}/\partial x = 0 \tag{2.4}$$

$$L_5(u^{(j)}, v^{(j)}) = -L_6(w^{(j)}) \leftrightarrow u^{(j)} = 0, \quad L_7(u^{(j)}, v^{(j)}) = -L_8(w^{(j)}) \leftrightarrow v^{(j)} = 0 \tag{2.5}$$

where $u^{(-1)} = v^{(-1)} = 0$; $q^{(0)} = q$; $q^{(j)} = 0$ for $j \geq 1$.

According to the proposed scheme we first solve Eq. (2.2) with the pair of boundary conditions (2.4), whence we find $w^{(j)}$. This boundary-value problem, apart from the coefficients, is identical to the problem of the bending of plates lying on an elastic foundation in classical theory, where the elastic foundation opposes the rotation of the sections of the plate producing a resistance in the form of a moment load $Bc^2\partial w/\partial x$ and $Bc^2\partial w/\partial y$ distributed over the area. We then solve the plane problem of the theory of elasticity on an elastic foundation (2.3) and (2.5). The resistance forces of the elastic foundation are simulated by the terms $4Bu$ and $4Bv$ in the operators L_{22} and L_{33}. After finding $u^{(j)}$ and $v^{(j)}$ we put we put $\epsilon = 1$ and, summing the series (2.1), we obtain the required solution, provided the series converge.

The second version of the iterative process is obtained if in (1.1) we additionally introduce a small parameter ϵ into the operators L_{22} and L_{33} in front of the terms $4Bu$ and $4Bv$. In this case, at the jth iteration, we will solve the plane problem (2.3) and (2.5) without an elastic foundation.

The terms on the right-hand sides of (2.2)–(2.5) can be assumed to be additional fictitious loads, determined in the previous iteration.

Hence, the proposed approach enables us to split the initial problem into two simpler problems, for which there are well-developed numerical methods of solution and for which it becomes possible to use approved software.

We will now introduce the small parameter ϵ into the eigenvalue problem. In this case in Eqs (1.1) and the boundary conditions (1.2) $L_{12}(u)$ is replaced by $\epsilon L_{12}(u)$, $L_{13}(v)$ by $\epsilon L_{13}(v)-\lambda M_1(w)$, $L_2(u,v)$ by $\epsilon L_2(u,v)+\epsilon\lambda M_2(w)$ and $L_4(u,v)$ by $\epsilon L_4(u,v)$.

If

$$M_1(w) = \rho H w, \quad M_2(w) = 0, \quad \lambda = \omega^2$$

$$H = \sum_{i=1}^{3} h_i, \quad \rho = \frac{1}{H}\sum_{i=1}^{3} \rho_i h_i$$

where ρ_i is the density of the ith layer, and ω is the angular frequency of the free vibrations, we have the problem of the low-frequency mainly bending free vibrations of a three-layer plate, which is obtained after separation of the variables.

If $M_1(w) = -\nabla^2 w$, $M_2(w) = \partial w/\partial x$, $\lambda = N^0$, where N^0 are the compressive forces along the contour of the plate, we have a problem of stability.

Substituting the series (2.1) and

$$\lambda = \sum_{j=0}^{\infty} \epsilon^j \lambda^{(j)}$$

into the modified equations and boundary conditions, we obtain an iterative process, similar to that given by Eqs (2.2)–(2.5), with the sole difference that instead of Eq. (2.2) we have

$$L_{11}(w^{(j)}) - \lambda^{(0)} M_1(w^{(j)}) = -L_{12}(u^{(j-1)}) - L_{13}(v^{(j-1)}) + \sum_{i=1}^{j} \lambda^{(i)} M_1(w^{(j-i)}) \tag{2.6}$$

and instead of the boundary conditions (2.4) we have

$$L_1(w^{(j)}) = -L_2(u^{(j-1)}, v^{(j-1)}) - \sum_{i=0}^{j-1} \lambda^{(i)} M_2(w^{(j-1-i)}) \leftrightarrow w^{(j)} = 0$$

$$L_3(w^{(j)}) = -L_4(u^{(j-1)}, v^{(j-1)}) \leftrightarrow \partial w^{(j)}/\partial x = 0 \tag{2.7}$$

We put $w^{(j)} = w_1^{(j)} + w_2^{(j)}$, where $w_1^{(j)}$ is found from the solution of the homogeneous equation (2.6) with non-homogeneous boundary conditions (2.7). To find $w_2^{(j)}$ we have Eq. (2.6) with homogeneous boundary conditions (2.7). Taking into account the symmetry of the operator on the left-hand side of (2.6), after scalar multiplication of Eq. (2.6) by $w^{(0)}$ we obtain the following formula for the eigenvalues

$$\lambda^{(j)} = \frac{(w^{(0)}, L_{12}(u^{(j-1)}) + L_{13}(v^{(j-1)}))}{(w^{(0)}, w^{(0)})} - \sum_{i=1}^{j-1} \lambda^{(i)} \frac{(w^{(0)}, M_1(w^{(j-1-i)}))}{(w^{(0)}, w^{(0)})}. \tag{2.8}$$

The second iterative scheme has the same differences as in the equations of statics, and Eq. (2.8) holds for it.

3. Consider the static problem and the eigenvalue problem for a rectangular plate hinge supported along the contour.

The boundary conditions at $x = 0$ and $x = a$ have the form

$$w = 0, \quad L_3(w) + L_4(u, v) = 0, \quad L_5(u, v) + L_6(w) = 0, \quad v = 0 \tag{3.1}$$

The corresponding conditions on the contours $y = 0$ and $y = b$ are obtained by making the substitutions $u \rightleftarrows v$ and $x \rightleftarrows y$ in conditions (3.1).

We obtain the solution of the problem in the form of series (summation is carried out over all m and n)

$$w = \Sigma w_{mn} \sin(\alpha x) \sin(\beta y), \quad u = \Sigma u_{mn} \cos(\alpha x) \sin(\beta y), \quad v = \Sigma v_{mn} \sin(\alpha x) \cos(\beta y) \tag{3.2}$$

which identically satisfy the boundary conditions.

Here, it is necessary to represent the transverse load in the form

$$q = \Sigma q_{mn} \sin(\alpha x) \sin(\beta y), \quad \alpha = \pi m/a, \quad \beta = \pi n/b \tag{3.3}$$

After substituting (3.2) and (3.3) into (1.1) we obtain

$$\begin{gathered} w_{mn} = H_{mn}^{-1}(1 - \xi_{mn})^{-1} q_{mn}, \quad u_{mn} = ½ \alpha \gamma^{-2} (\xi_{mn}^{-1} - F_{mn})^{-1} \\ v_{mn} = \beta \alpha^{-1} u_{mn}, \quad H_{mn} = \bar{D} \gamma^4 + Bc^2 \gamma^2, \quad F_{mn} = Bc + A_3 \gamma^2 h/12 \\ \xi_{mn} = 2F_{mn} H_{mn}^{-1} (A^* + G_{mn})^{-1}, \quad G_{mn} = 2B\gamma^{-2}, \quad \gamma^2 = \alpha^2 + \beta^2 \end{gathered} \tag{3.4}$$

In the eigenvalue problem we obtain

$$\rho H \lambda_{mn} = H_{mn}(1 - F_{mn} \xi_{mn}) \tag{3.5}$$

Problems using recurrent relations (2.2)–(2.5) and (2.3), (2.5), (2.6) and (2.7) are solved in the same way. We can obtain the following estimates of the error of the iterative process.

In the case of the first scheme with two elastic foundations, we have for the kth approximation

$$\begin{gathered} \eta_w^{(k)} = \eta_u^{(k)} = \eta_v^{(k)} = [F_{mn} \xi_{mn}]^{k+1}, \quad \rho H \lambda_{mn}^{(0)} = H_{mn}, \\ \rho H \lambda_{mn}^{(1)} = -F_{mn} H_{mn} \xi_{mn}, \; \lambda_{mn}^{(k)} = 0 \end{gathered} \tag{3.6}$$

In the case of the second scheme with a single elastic foundation, we have

$$\begin{gathered} \eta_w^{(0)} = 1 - H_{mn}^{-2} (1 - \xi_{mn})^{-1}, \quad \eta_w^{(k)} = F_{mn} \xi_{mn} [\zeta_{mn}]^k, \; \eta_u^{(k)} = \eta_v^{(k)} = [\zeta_{mn}]^{k+1} \\ \eta_\lambda^{(k)} = -F_{mn} \xi_{mn} (1 - F_{mn} \xi_{mn})^{-1} [-G_{mn}/A^*]^k, \quad \zeta_{mn} = 2F_{mn}^2 H_{mn}^{-1}/A^* - G_{mn}/A^* \\ \eta_r^{(k)} = \frac{r_{mn} - s_r^{(k)}}{r_{mn}}, \quad s_r^{(k)} = \sum_{i=0}^{k} r_{mn}^{(i)} \; (r = u, v, w, \lambda) \end{gathered} \tag{3.7}$$

Note that the expressions in square brackets in (3.6) and (3.7) are considerably less than unity over a wide range of variation of the geometrical and physical parameters of three-layer plates [3], which indicates that the methods converge well.

We will consider a numerical example. Suppose $a = b = 1$ m, $q = \text{const} = q_0$, $E = 10^5$ MPa, $E_3 = 2.5 \times 10^3$ MPa, $h = 0.025$ m, $h_3 = 0.1$ m and $\nu = \nu_3 = 0.25$.

Below we give the results of calculations (the number of the column corresponds to the number of the approximation, beginning with the zeroth approximation).

For the first iterative scheme

$w_{\max}^{(j)} \times 10^4/q_0$	451	133	38	10
$u_{\max}^{(j)} \times 10^5/q_0$	256	72	23	8
$\rho H \lambda_{11}^{(j)}$ m^{-1}	3197	−854	0	

For the second iterative scheme

$w_{\max}^{(j)} \times 10^4/q_0$	451	182	−1						
$u_{\max}^{(j)} \times 10^5/q_0$	361	−2	0						
$\rho H \lambda_{11}^{(j)}$ m^{-1}	3197	−1173	438	−163	61	−23	9	−3	1

It follows from the numerical results obtained that the iterative scheme with two elastic foundations is preferable in problems of dynamics and stability, while the scheme with a single elastic foundation is preferable in problems of statics. This is confirmed by the results presented in the dissertation referred to in the abstract, where the iterative scheme with a single elastic foundation was used to make a static calculation of circular plates with a point support. Satisfactory results were obtained at the zeroth-first approximations.

It should be noted that a similar method was effectively used in [4] for shallow shells.

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