Direct Eulerian Formulation of Anisotropic Hyperelasticity

K. Y. Volokh
Faculty of Civil and Environmental Engineering,
Technion - Israel Institute of Technology,
Haifa 3200003, Israel
e-mail: cvolokh@technion.ac.il

Many soft materials and biological tissues comprise isotropic matrix reinforced by fibers in the characteristic directions. Hyperelastic constitutive equations for such materials are usually formulated in terms of a Lagrangian strain tensor referred to the initial configuration and Lagrangian structure tensors defining characteristic directions of anisotropy. Such equations are "pushed forward" to the current configuration. Obtained in this way, Eulerian constitutive equations are often favorable from both theoretical and computational standpoints. In the present note, we show that the described two-step procedure is not necessary, and anisotropic hyperelasticity can be introduced directly in terms of an Eulerian strain tensor and Eulerian structure tensors referring to the current configuration. The newly developed constitutive equation is applied to the particular case of the transverse isotropy for the sake of illustration. [DOI: 10.1115/1.4049077]

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Let $\mathbf{m}_0^{(i)}$ denote a unit vector in the ith characteristic material direction in the initial configuration Ω_0 defined at time t_0. Let $\mathbf{F}$ denote the deformation gradient between Ω_0 and the current material configuration Ω at time t. Then, the characteristic direction in the current configuration becomes

$$\mathbf{m}^{(i)} = \mathbf{F}\mathbf{m}_0^{(i)} \tag{1}$$

This is the Lagrangian description. Its Eulerian equivalent is obtained by differentiation with respect to time as follows:

$$\dot{\mathbf{m}}^{(i)} = \dot{\mathbf{F}}\mathbf{m}_0^{(i)} \tag{2}$$

Substituting $\mathbf{m}_0^{(i)} = \mathbf{F}^{-1}\mathbf{m}^{(i)}$ from Eq. (1) into Eq. (2), we obtain

$$\dot{\mathbf{m}}^{(i)} = \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{m}^{(i)} \tag{3}$$

Introducing the velocity $\mathbf{v}$ gradient

$$\mathbf{L} = \operatorname{grad}\mathbf{v} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{4}$$

we can rewrite Eq. (1) in the purely Eulerian form of the evolution equation

$$\dot{\mathbf{m}}^{(i)} = \mathbf{L}\mathbf{m}^{(i)} \tag{5}$$

Note that vector $\mathbf{m}^{(i)}$ can always be replaced by vector $-\mathbf{m}^{(i)}$ in the reverse direction to define the characteristic anisotropy. To get rid of this nonuniqueness, it is helpful to introduce the so-called structure (or fabric) tensor in the form

$$\mathbf{M}^{(i)} = \mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)} \tag{6}$$

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By using Eq. (5), we derive its evolution equation

$$\dot{\mathbf{M}}^{(i)} = \mathbf{L}\mathbf{M}^{(i)} + \mathbf{M}^{(i)}\mathbf{L}^{\mathrm{T}} \tag{7}$$

The left Cauchy–Green tensor defines strains in the Lagrangian description

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} \tag{8}$$

Differentiating Eq. (8) with respect to time, we get

$$\begin{aligned}\dot{\mathbf{B}} &= \dot{\mathbf{F}}\mathbf{F}^{\mathrm{T}} + \mathbf{F}\dot{\mathbf{F}}^{\mathrm{T}} \\ &= \dot{\mathbf{F}}(\mathbf{F}^{-1}\mathbf{F})\mathbf{F}^{\mathrm{T}} + \mathbf{F}(\mathbf{F}^{-1}\mathbf{F})^{\mathrm{T}}\dot{\mathbf{F}}^{\mathrm{T}} \\ &= (\dot{\mathbf{F}}\mathbf{F}^{-1})(\mathbf{F}\mathbf{F}^{\mathrm{T}}) + (\mathbf{F}\mathbf{F}^{\mathrm{T}})(\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^{\mathrm{T}}) \\ &= \mathbf{L}\mathbf{B} + \mathbf{B}\mathbf{L}^{\mathrm{T}}\end{aligned} \tag{9}$$

Thus, evolution equation

$$\dot{\mathbf{B}} = \mathbf{L}\mathbf{B} + \mathbf{B}\mathbf{L}^{\mathrm{T}} \tag{10}$$

defines the left Cauchy–Green tensor in the Eulerian description.

In summary, we have kinematic tensors for strain $\mathbf{B}$ and anisotropy $\mathbf{M}^{(i)}$, which are defined in the Eulerian description via evolution equations (10) and (7) accordingly.

We further assume that deformation is purely hyperelastic and the internal dissipation vanishes

$$D_{\mathrm{int}} = \boldsymbol{\sigma}:\mathbf{D} - \varrho\dot{w} = 0 \tag{11}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{D} = (\mathbf{L} + \mathbf{L}^{\mathrm{T}})/2$ is the symmetric deformation rate tensor, ϱ is the current mass density, and w is the specific free energy per unit mass.

Targeting the Eulerian description, we assume that the specific free energy w is a function of kinematic variables $\mathbf{B}$ and $\mathbf{M}^{(i)}$. Consequently, we calculate the rate of the free energy as follows:

$$\dot{w}(\mathbf{B}, \mathbf{M}^{(i)}) = \frac{\partial w}{\partial \mathbf{B}}:\dot{\mathbf{B}} + \sum_i \frac{\partial w}{\partial \mathbf{M}^{(i)}}:\dot{\mathbf{M}}^{(i)} \tag{12}$$

It cannot be overemphasized that the structure tensor $\mathbf{M}^{(i)} = \mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)}$ is a kinematic variable in the Eulerian description. In the Lagrangian description, the initial structure tensor $\mathbf{M}_0^{(i)} = \mathbf{m}_0^{(i)} \otimes \mathbf{m}_0^{(i)}$ is fixed and, because of that, it is not a kinematic variable.

Substitution of Eqs. (10) and (7) in Eq. (12) yields

$$\dot{w} = 2\mathbf{A}:\mathbf{L} \tag{13}$$

where

$$\mathbf{A} = \frac{\partial w}{\partial \mathbf{B}}\mathbf{B} + \sum_i \frac{\partial w}{\partial \mathbf{M}^{(i)}}\mathbf{M}^{(i)} \tag{14}$$

With account of Eq. (13), dissipation equation (11) reads

$$D_{\mathrm{int}} = \boldsymbol{\sigma}:\mathbf{D} - \varrho(\mathbf{A} + \mathbf{A}^{\mathrm{T}}):\mathbf{D} - \varrho(\mathbf{A} - \mathbf{A}^{\mathrm{T}}):\mathbf{W} = 0 \tag{15}$$

where $\mathbf{W} = (\mathbf{L} - \mathbf{L}^{\mathrm{T}})/2$ is the antisymmetric spin tensor.

Noting that components of $\mathbf{D}$ and $\mathbf{W}$ are independent and using Eq. (14), we require $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ and finally obtain the constitutive equation of anisotropic hyperelasticity as follows:

$$\boldsymbol{\sigma} = 2\varrho\left[\frac{\partial w}{\partial \mathbf{B}}\mathbf{B} + \sum_i \frac{\partial w}{\partial \mathbf{M}^{(i)}}\mathbf{M}^{(i)}\right] \tag{16}$$

By way of example, let us consider transverse isotropy with one characteristic direction $\mathbf{m}$, which is defined by the free energy $w(I_1, I_2, I_3, I_4, I_5)$ depending on five invariants

$$\begin{aligned} I_1 &= \mathbf{1}:\mathbf{B} \\ I_2 &= \frac{1}{2}(I_1^2 - \mathbf{1}:\mathbf{B}^2) \\ I_3 &= \det \mathbf{B} \\ I_4 &= \mathbf{1}:\mathbf{M} \\ I_5 &= \mathbf{B}:\mathbf{M} \end{aligned} \tag{17}$$

where $\mathbf{1}$ is the identity tensor and $\mathbf{M} = \mathbf{m} \otimes \mathbf{m}$.

Three first invariants are the principal ones, while the last two invariants correspond to the characteristic direction of anisotropy. We note again that in the Eulerian description, "anisotropic invariants" are generally functions of both $\mathbf{B}$ and $\mathbf{M}$.

We calculate the following nontrivial derivatives of invariants

$$\begin{aligned} \frac{\partial I_1}{\partial \mathbf{B}} &= \mathbf{1} \\ \frac{\partial I_2}{\partial \mathbf{B}} &= I_1\mathbf{1} - \mathbf{B} \\ \frac{\partial I_3}{\partial \mathbf{B}} &= I_3\mathbf{B}^{-1} \\ \frac{\partial I_4}{\partial \mathbf{M}} &= \mathbf{1} \\ \frac{\partial I_5}{\partial \mathbf{B}} &= \mathbf{M} \\ \frac{\partial I_5}{\partial \mathbf{M}} &= \mathbf{B} \end{aligned} \tag{18}$$

Then, we get

$$\frac{\partial w}{\partial \mathbf{B}} = \sum_n w_n \frac{\partial I_n}{\partial \mathbf{B}} = (w_1 + w_2 I_1)\mathbf{1} - w_2\mathbf{B} + w_3 I_3 \mathbf{B}^{-1} + w_5\mathbf{M} \tag{19}$$

and

$$\frac{\partial w}{\partial \mathbf{M}} = \sum_n w_n \frac{\partial I_n}{\partial \mathbf{M}} = w_4\mathbf{1} + w_5\mathbf{B} \tag{20}$$

where $w_n \equiv \partial w / \partial I_n$.

Substitution of Eqs. (19) and (20) in Eq. (16) yields (compared with Ref. [1])

$$\begin{aligned} \boldsymbol{\sigma} = 2\varrho\big[&(w_1 + w_2 I_1)\mathbf{B} - w_2\mathbf{B}^2 + w_3 I_3 \mathbf{1} \\ &+ w_4\mathbf{M} + w_5(\mathbf{MB} + \mathbf{BM})\big] \end{aligned} \tag{21}$$

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Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

The datasets generated and supporting the findings of this article are obtainable from the corresponding author upon reasonable request. The authors attest that all data for this study are included in the paper. No data, models, or code were generated or used for this paper.

Reference

[1] Volokh, K. Y., 2019, *Mechanics of Soft Materials*, 2nd ed., Springer, Singapore.