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Different Features of a Stability Problem of Underconstrained Structures

Stability problem of bearing pin-jointed assemblies, in which the number of equilibrium equations is greater than the equilibrium matrix rank (underconstrained structures), is investigated. Local and overall stability of initial and loaded states are discussed. Theoretical considerations are accompanied by numerical examples.

1 Introduction

Pin-jointed assemblies in which nodal displacements do not produce elongations of members are traditionally called *mechanisms* or *kinematic chains.* These assemblies cannot bear external load.

On the contrary pin-jointed assemblies in which nodal displacements produce elongations of members are traditionally called *structures.* These assemblies can bear external load.

A specific class of pin-jointed assemblies called *underconstrained structures* can bear external load even though displacements exist which do not produce elongations of members. (The classification presented above is valid for small displacements.)

Some interest in the theory of underconstrained structures arised lately (Tarnai, 1980; Vilnay, 1990; Calladine and Pellegrino, 1991; Kuznetsov, 1991) in spite of the fact that such structures are used in engineering practice for a long time: cable nets, *tensegric* structures, and so on. The linear analysis of underconstrained structures, where the features of underconstrained structures behavior are clarified, is investigated in Volokh and Vilnay (1997a). It is shown that the rigidity of underconstrained structures is provided by initial equilibrium state or prestressing in contrast to conventional structures in which the rigidity is provided by the elastic properties of the system. This difference is because of the lack of members (or constraints) of underconstrained structures as compared to conventional structures. (The term "underconstrained structures" is derived from the lack of constraints.)

The present paper emphasizes the problem of the analysis of stability of underconstrained structures.

2 Initial Overall Stability

Arbitrary assembly of pin-jointed bars is in equilibrium if the following statical conditions are satisfied:

$$\mathbf{A}_0\mathbf{P}_0 = \mathbf{Q}_0. \quad (1)$$

$\mathbf{A}_0$ is an m by n initial configuration equilibrium matrix, $\mathbf{P}_0$ is an n-dimensional vector of initial member forces; $\mathbf{Q}_0$ is an m-dimensional vector of initial external loads.

In the particular case in which $\mathbf{Q}_0 = \mathbf{0}$, the self-stress state for nontrivial solution of Eq. (1) is obtained.

For simplicity, initial elongations and displacements are defined as zeros, indicating, generally, reference origin but not the physical absence of deformations. In fact, the deformed configuration is considered as an initial one.

Under external load $\mathbf{Q}$ Eq. (1) has to be replaced by

$$(\mathbf{A}_0 + \mathbf{A})(\mathbf{P}_0 + \mathbf{P}) = \mathbf{Q}_0 + \mathbf{Q} \quad (2)$$

in which the values without indexes are increments of the corresponding initial values.

By using Eq. (1) and linearizing Eq. (2) the last one takes the form

$$\mathbf{A}_0\mathbf{P} + \mathbf{A}\mathbf{P}_0 = \mathbf{Q}. \quad (3)$$

By adding Hooke's law and kinematic equations,

$$\mathbf{P} = \mathbf{S}\Delta \quad (4)$$

$$\mathbf{A}_0^T\mathbf{U} = \Delta \quad (5)$$

where $\mathbf{S}$ is the uncoupled stiffness matrix and $\mathbf{U}$ is m-dimensional vector of nodal displacements and Δ is n-dimensional vector of member elongations, a closed system of Eqs. (3)–(5) of the unknown vectors $\mathbf{U}$, $\mathbf{P}$, Δ is obtained.

Since after linearization perturbated equilibrium matrix $\mathbf{A}$ elements depend linearly upon nodal displacements, the second term on the left-hand side of Eq. (3) takes the form

$$\mathbf{A}\mathbf{P}_0 = \mathbf{D}\mathbf{U}$$

$$\mathbf{D} \equiv \mathbf{D}(\mathbf{P}_0) \quad (6)$$

$\mathbf{D}$ is an m by m matrix where the elements are linear combinations of the initial member forces presented by vector $\mathbf{P}_0$ components.

By using Eqs. (4)–(6), Eq. (3) takes the form

$$\mathbf{K}\mathbf{U} = \mathbf{Q} \quad (7)$$

$$\mathbf{K} = \mathbf{A}_0\mathbf{S}\mathbf{A}_0^T + \mathbf{D} \quad (8)$$

$\mathbf{K}$ is an m by m stiffness matrix.

Matrix $\mathbf{K}$ must be positive definite in order to provide overall stability of the initial equilibrium state.

In the case of conventional fully constrained structures, the first term on the right-hand side of Eq. (8) is of rank m. Taking into account that the first term elements are significantly larger than the second ones it is possible to neglect the second term on the right-hand side of Eq. (8). Positive definiteness of $\mathbf{S}$ provides positive definiteness of $\mathbf{K}$ and so initial equilibrium state of fully constrained conventional structures is always stable.

In the case of nonconventional underconstrained structures, the situation is more subtle. This is because the second term on the right-hand side of Eq. (8) cannot be neglected since rank $(\mathbf{A}_0\mathbf{S}\mathbf{A}_0^T) < m$. Generally speaking, it is necessary to check positive definiteness of matrix $\mathbf{K}$ by using both terms of Eq. (8). This straightforward approach is rough and abundant. Simplification of the problem is reached in the following manner:

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Let the displacement vectors be resolved into two mutually orthogonal vectors

$$\mathbf{U} = \mathbf{U}^{k} + \mathbf{U}^{e}. \quad (9)$$

$\mathbf{U}^k$ are "kinematic" displacement vectors, which do not produce member elongations, and $\mathbf{U}^e$ are "elastic" displacement vectors, which do produce members elongations. It is obvious from the homogeneous Eq. (5) that "kinematic" displacement vectors are presented in the form

$$\mathbf{U}^{k} = z_1\mathbf{e}_1 + \ldots + z_{m-r}\mathbf{e}_{m-r} \quad (10)$$

or

$$\mathbf{U}^{k} = \mathbf{W}\mathbf{Z}$$

$$\mathbf{W} = \{\mathbf{e}_1, \ldots, \mathbf{e}_{m-r}\}, \quad \mathbf{Z} = \{z_1, \ldots, z_{m-r}\}^T. \quad (11)$$

$\mathbf{e}_i$ is the vector of matrix $\mathbf{A}_0^T$ nullspace, z_i is the arbitrary scalar, r is the matrix $\mathbf{A}_0^T$ rank.

The "elastic" displacement vector belongs to the orthogonal complement subspace of $\mathcal{R}^m$ and it takes the form

$$\mathbf{U}^{e} = z_{m-r+1}\mathbf{e}_{m-r+1} + \ldots + z_m\mathbf{e}_m \quad (12)$$

or

$$\mathbf{U}^{e} = \tilde{\mathbf{W}}\tilde{\mathbf{Z}}$$

$$\tilde{\mathbf{W}} = \{\mathbf{e}_{m-r+1}, \ldots, \mathbf{e}_m\}, \quad \tilde{\mathbf{Z}} = \{z_{m-r+1}, \ldots, z_m\}^T. \quad (13)$$

$\mathbf{e}_i$ is the vector of matrix $\mathbf{A}_0^T$ row space and z_i is an arbitrary scalar.

By substituting Eqs. (9), (11), and (13), Eq. (7) takes the form

$$\mathbf{A}_0\mathbf{S}\mathbf{A}_0^T\tilde{\mathbf{W}}\tilde{\mathbf{Z}} + \mathbf{D}\tilde{\mathbf{W}}\tilde{\mathbf{Z}} + \mathbf{D}\mathbf{W}\mathbf{Z} = \mathbf{Q}. \quad (14)$$

Arbitrary vectors of the null space and the row space may be presented in the form:

$$\mathbf{V} = \mathbf{W}\mathbf{Y} \quad (15)$$

$$\tilde{\mathbf{V}} = \tilde{\mathbf{W}}\tilde{\mathbf{Y}}$$

$$\mathbf{Y} = \{y_1, \ldots, y_{m-r}\}^T, \quad \tilde{\mathbf{Y}} = \{y_{m-r+1}, \ldots, y_m\}^T \quad (16)$$

Multiplying scalarly Eq. (14) by vectors $\mathbf{V}^T$, $\tilde{\mathbf{V}}^T$, it is possible to obtain correspondingly

$$\mathbf{Y}^T(\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}\tilde{\mathbf{Z}} + \mathbf{W}^T\mathbf{D}\mathbf{W}\mathbf{Z}) = \mathbf{Y}^T\mathbf{W}^T\mathbf{Q} \quad (17)$$

$$\tilde{\mathbf{Y}}^T(\tilde{\mathbf{W}}^T\mathbf{A}_0\mathbf{S}\mathbf{A}_0^T\tilde{\mathbf{W}}\tilde{\mathbf{Z}} + \tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}}\tilde{\mathbf{Z}} + \tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}\mathbf{Z}) = \tilde{\mathbf{Y}}^T\tilde{\mathbf{W}}^T\mathbf{Q}. \quad (18)$$

The physical sense of the projecting is the utilization of the principle of virtual work. Since Eqs. (17), (18) should be satisfied under arbitrary vectors $\mathbf{Y}$, $\tilde{\mathbf{Y}}$, the equations can be rewritten in the form

$$(\mathbf{K}^e + \tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}})\tilde{\mathbf{Z}} + \tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}\mathbf{Z} = \tilde{\mathbf{W}}^T\mathbf{Q} \quad (19)$$

$$\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}\tilde{\mathbf{Z}} + \mathbf{K}^k\mathbf{Z} = \mathbf{W}^T\mathbf{Q} \quad (20)$$

where

$$\mathbf{K}^e = \tilde{\mathbf{W}}^T\mathbf{A}_0\mathbf{S}\mathbf{A}_0^T\tilde{\mathbf{W}} \quad (21)$$

$$\mathbf{K}^k = \mathbf{W}^T\mathbf{D}\mathbf{W}. \quad (22)$$

After some transformations Eqs. (19), (20) take the form

$$\{\mathbf{K}^e + \tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}} - \tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}(\mathbf{K}^k)^{-1}\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}\}\tilde{\mathbf{Z}} = \{\tilde{\mathbf{W}}^T - \tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}(\mathbf{K}^k)^{-1}\mathbf{W}^T\}\mathbf{Q} \quad (23)$$

$$\{\mathbf{K}^k - \mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}(\mathbf{K}^e + \tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}\}\mathbf{Z} = \{\mathbf{W}^T - \mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}(\mathbf{K}^e + \tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{W}}^T\}\mathbf{Q}. \quad (24)$$

By using the estimate

$$\|\mathbf{K}^e\| \approx \|\mathbf{S}\| \gg \|\mathbf{K}^k\| \approx \|\mathbf{D}\| \approx \|\mathbf{P}_0\| \quad (25)$$

where norms indicate, for example, maximum absolute values of elements, and neglecting small values, Eqs. (23) and (24) take the form

$$\mathbf{K}^e\tilde{\mathbf{Z}} = \tilde{\mathbf{R}}\mathbf{Q} \quad (26)$$

$$\mathbf{K}^k\mathbf{Z} = \mathbf{R}\mathbf{Q} \quad (27)$$

where

$$\tilde{\mathbf{R}} = \tilde{\mathbf{W}}^T - \tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}(\mathbf{K}^k)^{-1}\mathbf{W}^T \quad (28)$$

$$\mathbf{R} = \mathbf{W}^T - \mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}(\mathbf{K}^e)^{-1}\tilde{\mathbf{W}}^T. \quad (29)$$

This means that $\mathbf{K}^k$ is an $m - r$ by $m - r$ "kinematic" stiffness matrix and $\mathbf{K}^e$ is an r by r "elastic" stiffness matrix.

The two matrices $\mathbf{K}^k$ and $\mathbf{K}^e$ must be positive definite to provide overall stability of initial equilibrium state. Matrix $\mathbf{K}^e$ is always positive definite due to its framework. It is implied that det $(\mathbf{K}^e + \tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}}) \neq 0$, otherwise the state is unstable.

Matrix $\mathbf{K}^k$ positive definiteness must be checked under initial configuration design.

Suppose that the initial forces are induced due to prestressing. In this case, solution of homogeneous Eq. (1) is presented in the form

$$\mathbf{P}_0 = t_1\mathbf{p}_1 + \ldots + t_{n-r}\mathbf{p}_{n-r} \quad (30)$$

t_i is an arbitrary scalar and $\mathbf{p}_i$ is the matrix $\mathbf{A}_0$ nullspace basis vector.

In this case the "kinematic" stiffness matrix takes the form

$$\mathbf{K}^k = t_1\mathbf{K}_1^k + \ldots + t_{n-r}\mathbf{K}_{n-r}^k \quad (31)$$

$$\mathbf{K}_i^k = \mathbf{W}^T\mathbf{D}(\mathbf{p}_i)\mathbf{W}. \quad (32)$$

If the set of parameters t_i leads to positive definiteness of matrix $\mathbf{K}^k$, initial overall stability is provided. Generally speaking, an appropriate set of the parameters could be found under the design of a structure with the help of an algorithm such as that of Calladine-Pellegrino (1991) (CP procedure). The idea of the above algorithm can be described briefly as follows:

For this purpose, it is necessary to maximize parameter e by varying t_i under constraints

$$\mathbf{b}_j^T\Big(\sum_{i=1}^{n-r} t_i\mathbf{K}_i^k\Big)\mathbf{b}_j \geq e \geq 0, \quad j = 1, \ldots l \quad (33)$$

$$t_i^- \leq t_i \leq t_i^+. \quad (34)$$

$\mathbf{b}_j$ is the vector of m-dimensional Euclidian space $\mathcal{R}^m$; t_i^-, t_i^+ are lower and upper boundaries for parameter t_i.

On first iteration, $\mathbf{b}_j s$ are selected as unit base vectors in $\mathcal{R}^m$ and $l = m$. Under replacing parameters t_i by subtraction of two positive values, the initial problem is nothing but linear programming problem. If its solution $\{t_1^{(1)}, \ldots, t_{n-r}^{(1)}\}$ leads to positive definite $\mathbf{K}^k$ (all eigenvalues are positive) then the procedure is finished. Otherwise, existing unit vectors $\mathbf{b}_j$ are completed by m new vectors which are eigenvectors of matrix:

$$\overset{(1)}{\mathbf{K}}{}^k = \sum_{i=1}^{n-r} t_i^{(1)}\mathbf{K}_i^k$$

and all the calculations are repeated.

The new solution of the linear programming problem $\{t_1^{(2)}, \ldots, t_{n-r}^{(2)}\}$ leads to a new matrix

$$\overset{(2)}{\mathbf{K}}{}^{k} = \sum_{i=1}^{n-r} t_i^{(2)} \mathbf{K}_i^k$$

in which its positive definiteness is checked and so on.

Finally two cases are possible: (a) where a suitable set of nonzero parameters t_i is found or (b) where the found set is comprised of zeros. The latter means that the structure is unstable and can not be stiffened by prestressing.

The assembly shown in Fig. 1 is comprised of 12 members which form a regular quadratic net in plane. Nodes A, B, C, D are of the same height. The initial equilibrium matrix is 12 × 12 but of rank 11. It is easy to find self-stress distributions which satisfy the homogeneous Eq. (1). The initial force projections on the x-axis of members 1–6 should be equal to each other in order to satisfy the nodal equilibrium equations. This argument is valid also for members 7–12 in the y direction. The signs of the force values must be different for the two families of members (1–6 and 7–12) in order to satisfy the nodal equilibrium equations in the z-direction. For example, members 1–6 are under tension and members 7–12 are under compression.

This means that the assembly possesses "kinematic" displacements and may be a mechanism or underconstrained structure. The latter takes place when the self-stress state is stable and matrix $\mathbf{K}^k$ is positive definite. It is easy to observe without any calculation that the assembly is not stable. If it were stable under a given distribution of self-stresses, the matrix $\mathbf{K}^k$ eigenvalues would be positive. Let us, for instance, change the signs of the initial member forces so that members 1–6 are under compression and members 7–12 are under tension. In this case, the matrix $\mathbf{K}^k$ eigenvalue signs are changed too and, consequently, the matrix becomes negative definite, which implies that the new state is unstable. This means that the stability depends upon the numeration of members: If tensioned members are numerated from 1 to 6, the assembly will be stable, otherwise, it will be unstable. This conclusion contradicts the physical sense since the concept of stability is invariant to the transformation of the notation. Finally, initial assumption of the stability of the initial state of the assembly is invalid, and the assembly is an unstable mechanism.

This example emphasizes that the existence of a self-stress state is not enough to indicate that the assembly is a stable structure.

Consider the assembly shown in Fig. 2 (Volokh and Vilnay, 1997b). This assembly was created by adding nodes E, F, G, H and members 13–22 to the previous one. The assembly possesses 24 degrees-of-freedom and 22 members. Consequently, the initial equilibrium matrix is of the 24 × 22 dimension. Again the assembly is a mechanism or an underconstrained structure since the equilibrium matrix rank is smaller than the number of degrees-of-freedom. The rank of the above equilibrium matrix is found to be 19.

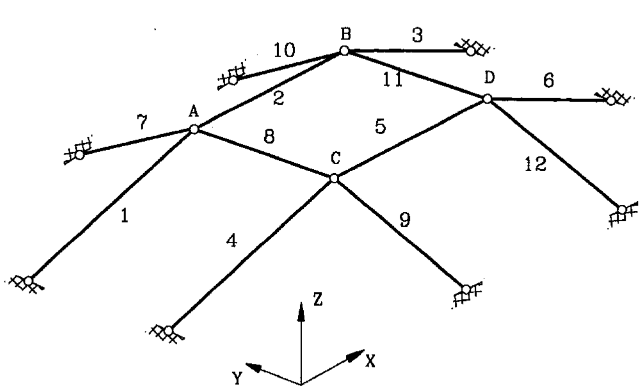

Fig. 1 12-bar assembly

Thus, the self-stress state is obtained with the accuracy of the three unknown parameters $\mathbf{t} = \{t_1, t_2, t_3\}$. These nonzero parameters may be obtained by the described CP-procedure to provide the initial overall stability. For example, in the case of the nodal heights,

$$h_A = h_B = h_C = h_D = 10, \quad h_E = h_F = h_G = h_H = 8.2,$$

the following distribution of the initial member forces leads to matrix $\mathbf{K}^k$ positive definiteness

$$P_{01} = P_{03} = P_{04} = P_{06} = 523.54$$
$$P_{02} = P_{05} = 468.28$$
$$P_{013} = P_{015} = P_{016} = P_{018} = 310.28$$
$$P_{014} = P_{017} = 287.03$$
$$P_{019} = P_{020} = P_{021} = P_{022} = 117.68$$
$$P_{07} = P_{09} = P_{010} = P_{012} = -786.68$$
$$P_{08} = P_{011} = -703.646 \qquad (*)$$

where minus indicates compression.

In the case of changing the heights of the nodes to

$$h_A = h_B = h_C = h_D = 20, \quad h_E = h_F = h_G = h_H = 18.2,$$

the new assembly will not be in equilibrium under the initial member forces given by formula (*). It is possible to change only some of the initial member forces, in order to equilibrate the assembly. If the member forces values are changed as follows,

$$P_{013} = P_{015} = P_{016} = P_{018} = 279.59$$
$$P_{014} = P_{017} = 258.64$$
$$P_{019} = P_{020} = P_{021} = P_{022} = 235.37,$$

and the rest remain the same, the structure will become unstable and it is necessary then to find another distribution of initial member forces to stabilize it.

As a whole, it is evident that the assembly shown in Fig. 2 can be stiffened by prestressing and can be defined as an underconstrained structure.

3 Loaded State Overall Stability

Instability of a loaded state can be interpreted in the traditional way; a bifurcation of the equilibrium problem. In the case of the bifurcation of the equilibrium state, nonzero perturbations $\mathbf{A}'$, $\mathbf{P}'$ exist,

$$(\mathbf{A}_0 + \mathbf{A} + \mathbf{A}')(\mathbf{P}_0 + \mathbf{P} + \mathbf{P}') = \mathbf{Q}_0 + \mathbf{Q}, \qquad (35)$$

by expanding Eq. (35) and omitting values of higher order

$$\mathbf{A}_0\mathbf{P}' + \mathbf{A}'(\mathbf{P}_0 + \mathbf{P}) = \mathbf{0} \qquad (36)$$

where

$$\mathbf{P}' = \mathbf{S}\mathbf{A}_0^{\mathrm{T}}\delta\mathbf{U}^{\mathrm{e}} \qquad (37)$$

$$\mathbf{A}'(\mathbf{P}_0 + \mathbf{P}) = \mathbf{D}'\delta(\mathbf{U}^{\mathrm{e}} + \mathbf{U}^{\mathrm{k}})$$
$$\mathbf{D}' \equiv \mathbf{D}(\mathbf{P}_0) + \mathbf{D}(\mathbf{P}). \qquad (38)$$

This means that equilibrium equations in terms of displacements can be obtained directly from Eqs. (19), (20) with the help of the following replacements:

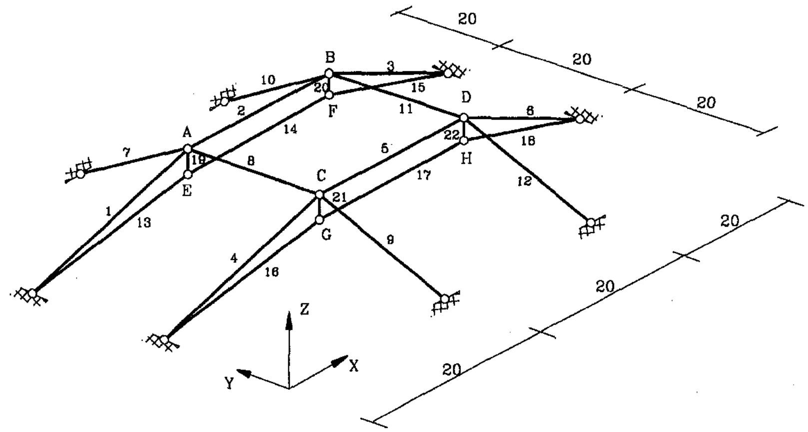

Fig. 2 22-bar assembly

$$\mathbf{Z} \to \delta\mathbf{Z},\ \tilde{\mathbf{Z}} \to \delta\tilde{\mathbf{Z}},\ \mathbf{D} \to \mathbf{D}',\ \mathbf{Q} \to \mathbf{0}$$

which take the form

$$(\mathbf{K}^e + \tilde{\mathbf{W}}^T\mathbf{D}'\tilde{\mathbf{W}})\delta\tilde{\mathbf{Z}} + \tilde{\mathbf{W}}^T\mathbf{D}'\mathbf{W}\,\delta\mathbf{Z} = \mathbf{0} \tag{39}$$

$$\mathbf{W}^T\mathbf{D}'\tilde{\mathbf{W}}\delta\tilde{\mathbf{Z}} + \hat{\mathbf{K}}^k\delta\mathbf{Z} = \mathbf{0} \tag{40}$$

where

$$\hat{\mathbf{K}}^k = \mathbf{W}^T\mathbf{D}'\mathbf{W}. \tag{41}$$

$\mathbf{P}$ is obtained from Eqs. (5), (4), (9), (13), (26),

$$\mathbf{P} = \mathbf{S}\mathbf{A}_0^T\mathbf{U}^e = \mathbf{S}\mathbf{A}_0^T\tilde{\mathbf{W}}(\mathbf{K}^e)^{-1}\tilde{\mathbf{R}}\mathbf{Q}. \tag{42}$$

In the case where the external load $\mathbf{Q}$ grows up proportionally by parameter ν, then member forces take the form $\nu\mathbf{P}$, and consequently

$$\mathbf{D}' \equiv \mathbf{D}(\mathbf{P}_0) + \nu\mathbf{D}(\mathbf{P}).$$

Nonzero displacements appear under such values of parameter ν, which lead to the following equality:

$$\det\begin{bmatrix} \mathbf{K}^e + \tilde{\mathbf{W}}^T\mathbf{D}'\tilde{\mathbf{W}} & \tilde{\mathbf{W}}^T\mathbf{D}'\mathbf{W} \\ \mathbf{W}^T\mathbf{D}'\tilde{\mathbf{W}} & \hat{\mathbf{K}}^k \end{bmatrix} = 0. \tag{43}$$

Estimate (25) indicates that the left upper submatrix elements are significantly larger than the rest. By using Laplace's resolution of the determinant, it is possible to replace approximately Eq. (43) by the following one:

$$\det(\mathbf{K}^e + \tilde{\mathbf{W}}^T\mathbf{D}'\tilde{\mathbf{W}})\det\hat{\mathbf{K}}^k = 0. \tag{44}$$

By using estimate (25) it is obvious that the minimum value of ν is obtained by using equation

$$\det\hat{\mathbf{K}}^k = 0 \tag{45}$$

or equation

$$\det\{\mathbf{K}^k + \nu\mathbf{W}^T\mathbf{D}(\mathbf{P})\mathbf{W}\} = 0. \tag{46}$$

Equation (46) represents the bifurcation of equilibrium problem of underconstrained structures.

It is interesting to compare this equation and the analogous one for conventional fully constrained structures

$$\det\{\mathbf{K} + \nu\mathbf{D}(\mathbf{P})\} = 0. \tag{47}$$

$\mathbf{K}$ is defined by Eq. (8).

The magnitudes of matrix $\mathbf{K}$ elements, which are determined by the elastic properties of the members, are significantly larger than the ones of matrix $\mathbf{K}^k$, which are determined by the values of the initial member forces only. As a consequence of this, the magnitude of the critical parameter ν_{cr} is determined by the elastic properties of the members in the case of fully constrained structures but not in the case of underconstrained structures. It is necessary to note that the increments of the member forces included in the second terms in the braces of Eqs. (46), (47) depend upon the correlations between the member stiffnesses and not upon the magnitude of their stiffnesses. In summary, the influence of the elastic properties of the members on the critical bifurcation load is negligible in the case of underconstrained structures which is in contrast to fully constrained ones.

The displacement modes at the bifurcation point are obtained in the following way: first, the nullspace of the matrix in the braces of Eq. (46) is calculated; this is no other but vector $\delta\mathbf{Z}$ with the accuracy of constant. Secondly, "kinematic" displacements are calculated with the help of Eq. (11) by replacing $\mathbf{Z}$ by $\delta\mathbf{Z}$. Vector $\delta\tilde{\mathbf{Z}}$ and the "elastic" displacements may be obtained from Eq. (39) but they are negligible, and so it is possible to claim (at least in an approximate manner) that the displacements modes at the bifurcation point are purely kinematic.

In the case of the underconstrained truss shown in Fig. 2, where the product of the Young's modulus and cross-section area is the same for all members,

$$E_i S_i = ES,$$

the critical values of the external loads for the four loading cases shown in Fig. 3 were calculated by using Eq. (46). The results take the form

$\nu_{cr} = 13.1342$ for case a;

$\nu_{cr} = 13.9680$ for case b;

$\nu_{cr} = 85.5169$ for case c;

$\nu_{cr} = 85.5169$ for case d.

The modes of displacement for the critical value of the load for case a are presented in Table 1 and Fig. 4.

In order to check the absence of the influence of the member elastic properties on the overall stability, the member stiffness was replaced significantly as follows:

$$E_i S_i \to l_i^3 ES.$$

Nevertheless, the new critical values of the external loads practically coincide with the previous ones

$\nu_{cr} = 13.4986$ for case a;

$\nu_{cr} = 12.9478$ for case b;

$\nu_{cr} = 86.4936$ for case c;

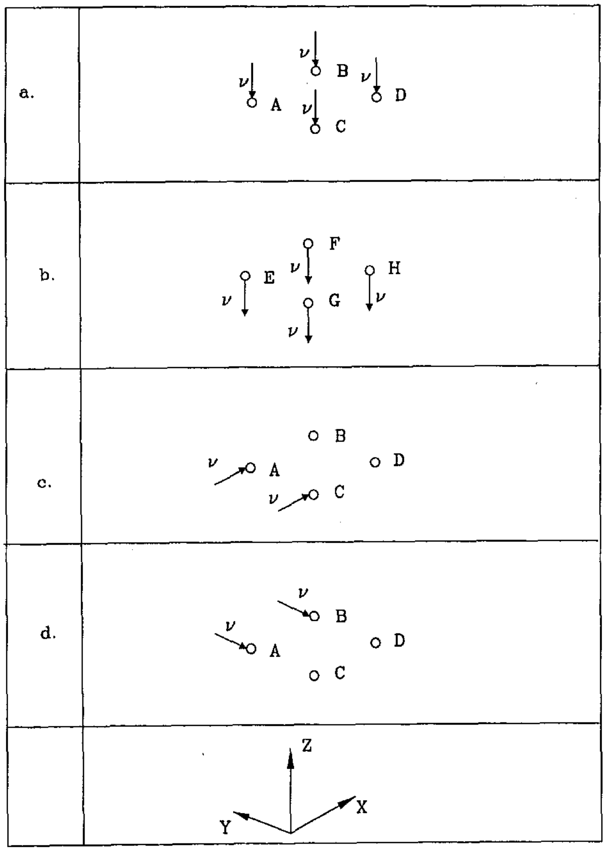

Fig. 3 Cases of loading

$$\nu_{cr} = 86.4936 \quad \text{for case d.}$$

4 Initial and Loaded States Local Stability

A local stability problem may be formulated as the loss of stability of one member which unconditionally leads to a general collapse of an underconstrained structure. Figure 5 and the following arguments show that the local stability problem can be formulated as a classical Euler's bifurcation problem for a hingely supported column. Let us cut member i with member force P_i with ajacent members as shown in Fig. 5(a). This fragment of the structure must be in equilibrium. To replace action of the adjacent members on the nodes by summary load, it is necessary to consider the equilibrium of the nodes (Fig. 5(b)). Evidently, the equilibrium of the nodes is satisfied where the sums of the adjacent member forces equal the member force P_i in the opposite direction. By these means the ith member is under external load P_i (Fig. 5(c)) and, consequently, Euler's classical problem takes a place (Fig. 5(d)). Thus, stability of the ith member is defined by

Table 1

Nodes/displacement direction	x	y	z
A	0.15	−0.15	−0.30
B	0.15	0.15	0.30
C	−0.15	−0.15	0.30
D	−0.15	0.15	−0.30
E	0.12	−0.09	−0.30
F	0.12	0.09	0.30
G	−0.12	−0.09	0.30
H	−0.12	0.09	−0.30

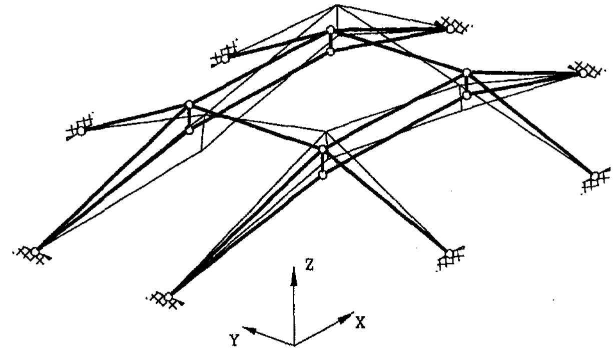

Fig. 4 Buckling mode

$$P_i = \frac{\pi^2 E_i I_i}{l_i^2}.$$

Since P_i may be due to prestressing or external load, this approach to local stability does not differ for initial and loaded states.

In the case of the assembly shown in Fig. 2, where dimensions are given in meters, the prestressing forces calculated by using formulae (*) are given in kilograms and the compressed members 7, 9, 10, 12 are made of steel with circular cross sections, the moment of inertia takes the form

$$I = I_7 = I_9 = I_{10} = I_{12} = \frac{\pi r^4}{4}$$

where r is the radius of the cross sections. In the case where Young's modulus is the same to all members,

$$E = E_7 = E_9 = E_{10} = E_{12} = 2.1 \cdot 10^6 \text{ kg/cm}^2,$$

it is possible to obtain a critical radius to the members, taking into account that the member length is 2236 cm

$$r^{cr} = \sqrt[4]{\frac{786.68 \cdot 4 \cdot 2236^2}{\pi^3 \cdot 2.1 \cdot 10^6}} \cong 4 \text{ cm}.$$

5 Conclusions

1 The stability of the initial state of underconstrained structures is not provided automatically. The formal existence of a self-stress state does not indicate that the structure is stable. This is contrary to the conventional fully constrained structures.

2 Overall stability of loaded underconstrained structures is affected by the initial equilibrium state or prestressing, not by the elastic properties of the system as in conventional structures.

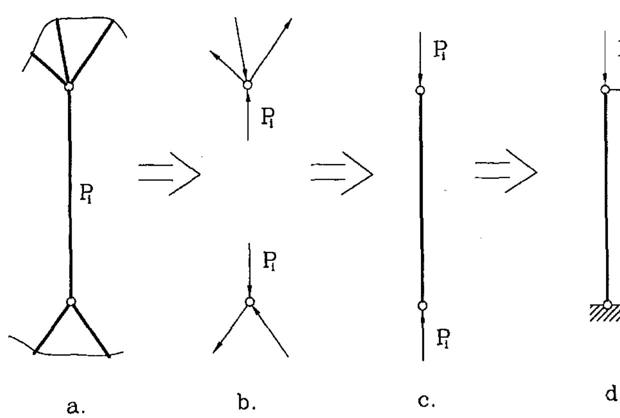

Fig. 5 On the formulation of the local stability problem

3 Local stability of underconstrained structures is affected by elastic properties of members and can be formulated in the classical Euler's manner.

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