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Nonlinear analysis of underconstrained structures

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Abstract

Underconstrained (kinematically indeterminate) assemblies of bars and pin-joints possess a specific kind of nonlinearity which appears even for small displacements and linear elasticity. Various approaches to nonlinear analysis based on the Newton–Raphson procedure are considered. The subspace Newton–Raphson technique is proposed. Theoretical considerations are accompanied by numerical examples of plane and space underconstrained assemblies.

1. Introduction

The concept of statical determinacy and indeterminacy is familiar to structural engineers. Degree of statical indeterminacy equals the difference between the number of the members of the assembly and the rank of the equilibrium matrix. Similarly a concept of kinematic determinacy and indeterminacy may be introduced. The degree of kinematic indeterminacy equals the difference between the number of degrees of freedom of the assembly and the rank of the geometric (transposed equilibrium) matrix. Nonzero degree of kinematic indeterminacy means existence of infinitesimal displacements which do not produce elongations of the members or, in other words, it means that the system of linear homogeneous kinematic equations possesses a nontrivial solution (infinitesimal mechanism).

Usually, kinematically indeterminate assemblies (mechanisms or kinematic chains) cannot bear an external load. However, there is a specific class of kinematically indeterminate assemblies—underconstrained structures, which can bear an external load. These underconstrained structures are of practical and theoretical interest. On the one hand, they are very light and their use leads to economy of materials. On the other hand, a theory of underconstrained structures allows to complete, in some sense, classical structural mechanics of pin-jointed assemblies. Interest in the theory of underconstrained structures arose lately (Calladine and Pellegrino, 1991; Kuznetsov, 1991; Tarnai, 1980; Vilnay, 1990); although they were used for a long time in engineering practice: underconstrained cable nets, tensegrity structures and so on.

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Problems of design and linear analysis of underconstrained structures based on the displacement method and concept of stiffness matrix were developed in Volokh and Vilnay (1997a, b). Analysis of underconstrained structures which takes into account geometrical nonlinearity is considered below.

2. Newton–Raphson procedures

2.1. Formulation of the problem

Mathematically, analysis of underconstrained (as well as fully constrained) structures is based on:

- equilibrium equations

$$\mathbf{B}^T(\mathbf{P}_0+\mathbf{P}) = \mathbf{Q} \tag{1}$$

where $\mathbf{B}$ is an $n \times m$ geometric matrix of direction cosines; $\mathbf{P}_0$ and $\mathbf{P}$ are vectors of initial member forces and force increments; $\mathbf{Q}$ is a vector of external nodal loads;

- constitutive equations (Hooke's law)

$$\mathbf{P} = \mathbf{S}\Delta \tag{2}$$

where $\mathbf{S}$ is an $n \times n$ uncoupled stiffness matrix with diagonal nonzero entries: $S_i = E_iF_i/l_i$ including the ith member Young modulus, cross-section area and length correspondingly; Δ is a vector of member elongations;

- kinematic equations

$$\Delta = \Delta(\mathbf{U}) \tag{3}$$

which present member elongations as a function of nodal displacements.

Various forms of eqn (3) are possible and will be discussed below (Section 4). It is important, however, that the choice of kinematic equations influences the form of equilibrium equations (matrix $\mathbf{B}$). This relationship may be traced, generally, by using the principle of virtual displacements or, in the case of elasticity, with the help of the principle of stationarity of potential energy. This is the only way to obtain equilibrium equations consistent with used kinematic assumptions.

Let the internal energy of deformation be written in matrix and componentwise forms as follows

$$\Omega = \tfrac{1}{2}\Delta^T\mathbf{S}\Delta + \mathbf{P}_0^T\Delta \tag{4}$$

$$\Omega = \sum_{i=1}^{n}\left(\frac{1}{2}S_i\Delta_i^2 + P_{0i}\Delta_i\right) \tag{4'}$$

then the left hand side of eqn (1) is obtained by differentiating eqn (4) with respect to displacements

$$\frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{U}} = \left(\frac{\partial \boldsymbol{\Delta}}{\partial \mathbf{U}}\right)^T (\mathbf{P}_0 + \mathbf{P}) \tag{5}$$

$$\frac{\partial \Omega}{\partial U_j} = \sum_{i=1}^{n} \left(S_i \Delta_i \frac{\partial \Delta_i}{\partial U_j} + P_{0i} \frac{\partial \Delta_i}{\partial U_j} \right) = \sum_{i=1}^{n} \frac{\partial \Delta_i}{\partial U_j} (P_{0i} + P_i) \tag{5$'$}$$

Thus,

$$\mathbf{B} = \frac{\partial \boldsymbol{\Delta}}{\partial \mathbf{U}} \tag{6}$$

$$B_{ij} = \frac{\partial \Delta_i}{\partial U_j} \tag{6$'$}$$

2.2. Newton–Raphson procedure based on purely displacement formulation of the problem (*DNR*)

By substituting eqns (2), (3) into eqn (1) it is possible to obtain equilibrium equations in terms of displacements. The DNR procedure takes the following form in this case

$$\overset{i}{\mathbf{K}}\mathbf{d}\overset{i}{\mathbf{U}} = \mathbf{Q} - \overset{i}{\mathbf{B}}{}^T(\mathbf{P}_0 + \overset{i}{\mathbf{P}}) \tag{7}$$

$$\overset{i}{\mathbf{P}} = \mathbf{S}\overset{i}{\boldsymbol{\Delta}} \tag{8}$$

$$\overset{i+1}{\mathbf{U}} = \overset{i}{\mathbf{U}} + \mathbf{d}\overset{i}{\mathbf{U}} \tag{9}$$

$$\overset{0}{\mathbf{U}} = \mathbf{0} \tag{10}$$

with the upper index designating iteration number.

The tangent stiffness matrix may be written as follows

$$\overset{i}{\mathbf{K}} = \left.\frac{\partial^2 \boldsymbol{\Omega}}{\partial \mathbf{U}\, \partial \mathbf{U}}\right|_{\overset{i}{\mathbf{U}}} = \overset{i}{\mathbf{A}} + \overset{i}{\mathbf{D}} \tag{11}$$

$$\overset{i}{K}_{jk} = \left.\frac{\partial^2 \Omega}{\partial U_k\, \partial U_j}\right|_{\overset{i}{\mathbf{U}}} = \overset{i}{A}_{jk} + \overset{i}{D}_{jk} \tag{11$'$}$$

where

$$\overset{i}{\mathbf{A}} = \mathbf{B}^T\mathbf{S}\mathbf{B}|_{\overset{i}{\mathbf{U}}} \tag{12}$$

$$\overset{i}{A}_{jk} = \left.\sum_{l=1}^{n} \frac{\partial \Delta_l}{\partial U_j} \frac{\partial \Delta_l}{\partial U_k} S_l \right|_{\overset{i}{\mathbf{U}}} \tag{12$'$}$$

$$\overset{i}{\mathbf{D}} = \left.\frac{\partial [\mathbf{B}^T(\mathbf{P}_0 + \mathbf{P})]}{\partial \mathbf{U}}\right|_{\overset{i}{\mathbf{U}}} \tag{13}$$

$$\overset{i}{D}_{jk} = \sum_{l=1}^{n} \frac{\partial^2 \Delta_l}{\partial U_j \, \partial U_k}(P_{0l} + P_l)\Big|_{\overset{i}{\mathbf{U}}} \tag{13'}$$

By designating Euclidean vector norm as $\|\cdot\|_2$ and tolerance as β it is possible to formulate the convergence criterion in the following form

$$\|\mathbf{Q} - \overset{i}{\mathbf{B}}{}^T(\mathbf{P}_0 + \overset{i}{\mathbf{P}})\|_2 \|\mathbf{Q}\|_2^{-1} \leqslant \beta \sim 0.01 \div 0.001 \tag{14}$$

2.3. Newton–Raphson procedure based on force–displacement formulation of the problem (FDNR)

In this case both displacements and force increments are considered as unknowns. Substitution of eqn (3) into eqn (2) leads to the following coupled system of equations

$$\begin{aligned} \mathbf{B}^T(\mathbf{P}_0 + \mathbf{P}) &= \mathbf{Q} \\ \mathbf{S}\Delta - \mathbf{P} &= \mathbf{0} \end{aligned} \tag{15}$$

Applying the NR procedure to these equations we obtain

$$\begin{aligned} \overset{i}{\mathbf{B}}{}^T\mathbf{d}\overset{i}{\mathbf{P}} + \overset{i}{\mathbf{D}}\mathbf{d}\overset{i}{\mathbf{U}} &= \mathbf{Q} - \overset{i}{\mathbf{B}}{}^T(\mathbf{P}_0 + \overset{i}{\mathbf{P}}) \\ \mathbf{S}\overset{i}{\mathbf{B}}\mathbf{d}\overset{i}{\mathbf{U}} - \mathbf{d}\overset{i}{\mathbf{P}} &= \overset{i}{\mathbf{P}} - \mathbf{S}\overset{i}{\Delta} \end{aligned} \tag{16}$$

Excluding $\mathbf{d}\overset{i}{\mathbf{P}}$ from the first equation the following is obtained

$$\overset{i}{\mathbf{K}}\mathbf{d}\overset{i}{\mathbf{U}} = \mathbf{Q} - \overset{i}{\mathbf{B}}{}^T(\mathbf{P}_0 + \mathbf{S}\overset{i}{\Delta}) \tag{17}$$

$$\overset{i+1}{\mathbf{P}} = \overset{i}{\mathbf{P}} + \mathbf{d}\overset{i}{\mathbf{P}} = \mathbf{S}\overset{i}{\Delta} + \mathbf{S}\overset{i}{\mathbf{B}}\mathbf{d}\overset{i}{\mathbf{U}} \tag{18}$$

$$\overset{i+1}{\mathbf{U}} = \overset{i}{\mathbf{U}} + \mathbf{d}\overset{i}{\mathbf{U}} \tag{19}$$

$$\overset{0}{\mathbf{U}} = \mathbf{0}; \quad \overset{0}{\mathbf{P}} = \mathbf{0} \tag{20}$$

The convergence criterion takes the form

$$\|\mathbf{Q} - \overset{i}{\mathbf{B}}{}^T(\mathbf{P}_0 + \mathbf{S}\overset{i}{\Delta})\|_2 \|\mathbf{Q}\|_2^{-1} \leqslant \beta$$

3. Orthogonal decomposition of displacements and "subspace" Newton–Raphson procedures

In this section it is considered how to take into account the specific features of underconstrained assemblies' behaviour and to modify nonlinear analysis accordingly. The main idea may be traced to linear analysis. In this case the NR procedure is limited by a single iteration

$$(\overset{0}{\mathbf{B}}{}^T\mathbf{S}\overset{0}{\mathbf{B}} + \overset{0}{\mathbf{D}})\mathbf{U} = \mathbf{Q}$$

which presents the key equation of linear analysis. Matrix $\overset{0}{\mathbf{B}}{}^T\mathbf{S}\overset{0}{\mathbf{B}}$ is dominant but rank deficient.

The latter is the reason why $\overset{0}{\mathbf{D}}$ cannot be neglected in the case of underconstrained structures as contrasted to fully constrained ones. The structure of the initial stiffness matrix suggests special orthogonal decomposition of displacements and subsequent transformation of the equation: Volokh and Vilnay (1997a). It is possible to extend this technique to nonlinear analysis assuming that displacements are small and, consequently, the structure of the tangent stiffness matrix is similar to its initial form.

The displacement vector is presented as a sum of two mutually orthogonal vectors

$$\mathbf{U} = \mathbf{U}^e + \mathbf{U}^k \tag{21}$$

$$\mathbf{U}^k = Z_1\mathbf{e}_1 + \cdots + Z_{m-r}\mathbf{e}_{m-r} \tag{22}$$

$$\mathbf{U}^e = Z_{m-r+1}\mathbf{e}_{m-r+1} + \cdots + Z_m\mathbf{e}_m \tag{23}$$

or

$$\mathbf{U}^k = \mathbf{W}\mathbf{Z}; \quad \mathbf{W} = \{\mathbf{e}_1, \ldots, \mathbf{e}_{m-r}\}; \quad \mathbf{Z} = \{Z_1, \ldots, Z_{m-r}\}^T \tag{24}$$

$$\mathbf{U}^e = \tilde{\mathbf{W}}\tilde{\mathbf{Z}}; \quad \tilde{\mathbf{W}} = \{\mathbf{e}_{m-r+1}, \ldots, \mathbf{e}_m\}; \quad \tilde{\mathbf{Z}} = \{Z_{m-r+1}, \ldots, Z_m\}^T \tag{25}$$

Here $\mathbf{Z}$, $\tilde{\mathbf{Z}}$ are vectors of new unknowns; columns of matrices $\mathbf{W}$ and $\tilde{\mathbf{W}}$ form orthonormal bases of the nullspace and row space of matrix $\overset{0}{\mathbf{B}}$ correspondingly; r is the rank of $\overset{0}{\mathbf{B}}$ and $m-r$ is nothing but the degree of kinematic indeterminacy. It is easy to observe the physical meaning of the decomposition of displacements given by formulae (22)–(25). Vector $\mathbf{U}^k$ (k for "kinematic") is solution of homogeneous initial kinematic equations. This means that it presents infinitesimal displacements which do not produce member elongations or, more accurately, the elongations are of smaller order of magnitude than displacements. On the contrary, vector $\mathbf{U}^e$ (e for "elastic") represents displacements which produce member elongations of the same order of magnitude as displacements. Now displacement increments take the form

$$\mathbf{dU} = [\tilde{\mathbf{W}}\mathbf{W}] \begin{bmatrix} \mathbf{d}\tilde{\mathbf{Z}} \\ \mathbf{dZ} \end{bmatrix} \tag{26}$$

By substituting eqn (26) into eqn (7) or (17) and premultiplying the latter by $[\tilde{\mathbf{W}}\mathbf{W}]^T$ from the left it is possible to obtain

$$\begin{bmatrix} \overset{i}{\mathbf{K}}{}^e & \overset{i}{\mathbf{L}} \\ \overset{i}{\mathbf{L}}{}^T & \overset{i}{\mathbf{K}}{}^k \end{bmatrix} \begin{bmatrix} \mathbf{d}\overset{i}{\tilde{\mathbf{Z}}} \\ \mathbf{d}\overset{i}{\mathbf{Z}} \end{bmatrix} = \begin{bmatrix} \overset{i}{\mathbf{Q}}_1 \\ \overset{i}{\mathbf{Q}}_2 \end{bmatrix} \tag{27}$$

where $\overset{i}{\mathbf{Q}}_1$, $\overset{i}{\mathbf{Q}}_2$ are right hand side of eqn (7) or (17) premultiplied by $\tilde{\mathbf{W}}^T$ and $\mathbf{W}^T$ correspondingly; $\mathbf{K}^e = \tilde{\mathbf{W}}^T\mathbf{K}\tilde{\mathbf{W}}$ is an $r \times r$ elastic tangent stiffness matrix; $\mathbf{K}^k = \mathbf{W}^T\mathbf{K}\mathbf{W}$ is an $m-r \times m-r$ kinematic tangent stiffness matrix; $\mathbf{L} = \tilde{\mathbf{W}}^T\mathbf{K}\mathbf{W}$.

Equation (27) may be rewritten as follows:

$$[\overset{i}{\mathbf{K}}{}^k - \overset{i}{\mathbf{L}}{}^T(\overset{i}{\mathbf{K}}{}^e)^{-1}\overset{i}{\mathbf{L}}]\mathbf{d}\overset{i}{\mathbf{Z}} = \overset{i}{\mathbf{Q}}_2 - \overset{i}{\mathbf{L}}{}^T(\overset{i}{\mathbf{K}}{}^e)^{-1}\overset{i}{\mathbf{Q}}_1 \tag{28}$$

$$\overset{i}{\mathbf{K}}{}^{e}\overset{i}{\mathbf{d}\tilde{\mathbf{Z}}} = \overset{i}{\mathbf{Q}}_1 - \overset{i}{\mathbf{L}}\overset{i}{\mathbf{d}\mathbf{Z}} \tag{29}$$

Neglecting small values in the above equations we obtain

$$\overset{i}{\mathbf{K}}{}^{k}\overset{i}{\mathbf{d}\mathbf{Z}} = \overset{i}{\mathbf{Q}}_2 \tag{30}$$

$$\overset{0}{\mathbf{K}}{}^{e}\overset{i}{\mathbf{d}\tilde{\mathbf{Z}}} = \overset{i}{\mathbf{Q}}_1 - \overset{i}{\mathbf{L}}\overset{i}{\mathbf{d}\mathbf{Z}} \tag{31}$$

Thus the kinematic stiffness matrix of small dimension ($m-r \ll m$) is inverted at every iteration, while the elastic stiffness matrix of large dimensions is constant and must be inverted only once. Taking into account that convergence of the procedure is affected mainly by kinematic displacements which span only the $m-r$ subspace of the displacement space it is natural to call this procedure "subspace". Implementation of eqns (30), (31) instead of eqn (7) leads to the subspace NR procedure on the base of displacement formulation (SDNR). Implementation of eqns (30), (31) instead of eqn (17) leads to the subspace NR procedure on the base of force–displacement formulation (SFDNR).

An important feature of the subspace technique is the possibility to identify the case of "equilibrium load" (which lies in the column space of the initial equilibrium matrix $\overset{0}{\mathbf{B}}{}^{T}$) at the first iteration. In this case

$$\|\overset{1}{\mathbf{U}}{}^{k}\|_2 \|\overset{1}{\mathbf{U}}{}^{e}\|_2^{-1} \leqslant 1 \tag{32}$$

and the procedure may be interrupted as unnecessary.

4. Kinematic equations

All the above considerations were carried out without explicit formulation of kinematic eqns (3). Let now various forms of these equations be considered.

4.1. Exact equations

This is the most obvious case which does not use any assumption

$$\Delta_i = l_i' - l_i \tag{33}$$

$$l_i' = \sqrt{(X_j + U_j - X_s - U_s)^2 + (X_{j-1} + U_{j+1} + X_{s+1} - U_{s+1})^2 + (X_{j+2} + U_{j+2} - X_{s+2} - U_{s+2})^2} \tag{34}$$

$$l_i = \sqrt{(X_j - X_s)^2 + (X_{j+1} - X_{s+1})^2 + (X_{j+2} - X_{s+2})^2} \tag{35}$$

It should be noted that U_jth displacement is zeroed if it corresponds to a supporting point.

Thus the geometric matrix takes the following form

$$B_{ij} = \frac{\partial \Delta_i}{\partial U_j} = \frac{X_j - X_s + U_j - U_s}{l_i'} \tag{36}$$

Entries of the geometric stiffness matrix are exact direction cosines for deformed configuration.

4.2. Small strains

By introducing new notation

$$\varepsilon_i = e_i + \omega_i \tag{37}$$

$$e_i = \frac{X_j - X_s}{l_i} \frac{U_j - U_s}{l_i} + \frac{X_{j+1} - X_{s+1}}{l_i} \frac{U_{j+1} - U_{s+1}}{l_i} + \frac{X_{j+2} - X_{s+2}}{l_i} \frac{U_{j+2} - U_{s+2}}{l_i} \tag{38}$$

$$\omega_i = \frac{1}{2}\left(\frac{U_j - U_s}{l_i}\right)^2 + \frac{1}{2}\left(\frac{U_{j+1} - U_{s+1}}{l_i}\right)^2 + \frac{1}{2}\left(\frac{U_{j+2} - U_{s+2}}{l_i}\right)^2 \tag{39}$$

eqn (34) takes the form

$$l_i' = l_i\sqrt{1 + 2\varepsilon_i} \tag{40}$$

On the other hand an "engineering" strain or relative elongation of the ith member is defined as follows

$$T_i = \frac{\Delta_i}{l_i} = \frac{l_i' - l_i}{l_i} \tag{41}$$

from which it is obtained that

$$l_i' = l_i(1 + T_i) \tag{42}$$

By equating right hand sides of eqns (40) and (42) it is possible to conclude that

$$T_i + \tfrac{1}{2} T_i^2 = \varepsilon_i \tag{43}$$

In case of small strains (but not displacements) eqn (43) is replaced approximately by the following

$$T_i = \varepsilon_i \ll 1 \tag{44}$$

Thus the ith member elongation takes the form $\Delta_i = l_i \varepsilon_i$ and with account of eqns (37)–(39) the geometric matrix takes the following form

$$B_{ij} = \frac{\partial \Delta_i}{\partial U_j} = \frac{X_j - X_s + U_j - U_s}{l_i} \tag{45}$$

Intuitively, this equation may be obtained directly from eqn (36) in assumption $l_i' = l_i$.

4.3. Small displacements

In this case we assume that

$$\alpha = \frac{U_j}{l_i} \ll 1 \tag{46}$$

for appropriate i and j and, expanding $\Delta_i = l_i(\sqrt{1+2\varepsilon_i}-1)$ into power series about displacements, obtain

$$\Delta_i = l_i\left(\varepsilon_i - \frac{e_i^2}{2}\right) \tag{47}$$

Terms of the third- and higher-orders of magnitude with respect to α are omitted. The ijth entry of the geometric matrix is

$$B_{ij} = \frac{\partial \Delta_i}{\partial U_j} = \frac{(X_j - X_s)(1-e_i) + U_j - U_s}{l_i} \tag{48}$$

4.4. Discussion

Comparing entries of geometric matrices presented by eqns (36), (45) and (48) in accordance with exact kinematics, small strains and small displacement assumptions, it is possible to conclude that the assumption of small strains leads to the simplest and computationally preferable scheme, which is also general enough for linear elasticity. However, the final choice of kinematics should be left to numerical examples which allow to compare convergence of the procedures based on different kinematics.

Another interesting and, mainly, theoretical aspect of comparison of various formulations of kinematics is the possibility to identify "smallness" of displacements. Indeed, let some computed displacements satisfy estimate $\alpha \sim 0.1$ or 0.01 or 0.001. Are they small? This question may be answered only by comparing results based on kinematics with and without small displacement assumptions.

5. Numerical examples

Two structures are considered. The first one (Fig. 1) is a plane underconstrained cable net and the second one (Fig. 2) is an underconstrained space assembly symmetric relatively horizontal middle plane. Both structures comprise members of circular cross-section diameter 0.4 cm and elasticity modulus $2.1 \cdot 10^6$ kg/cm^2. The plane net includes 11 members and possesses 12 degrees of freedom. Its degree of kinematic indeterminacy is two. The space assembly includes 145 members and possesses 150 degrees of freedom. Its degree of kinematic indeterminacy is 15. Pre-stressing forces of both structures are given in the first column of Tables 1 and 2. Pre-stressing forces of the space assembly are given for some typical members only and may be extended to the rest with account of symmetry. The second and the third columns of the tables present computed values of

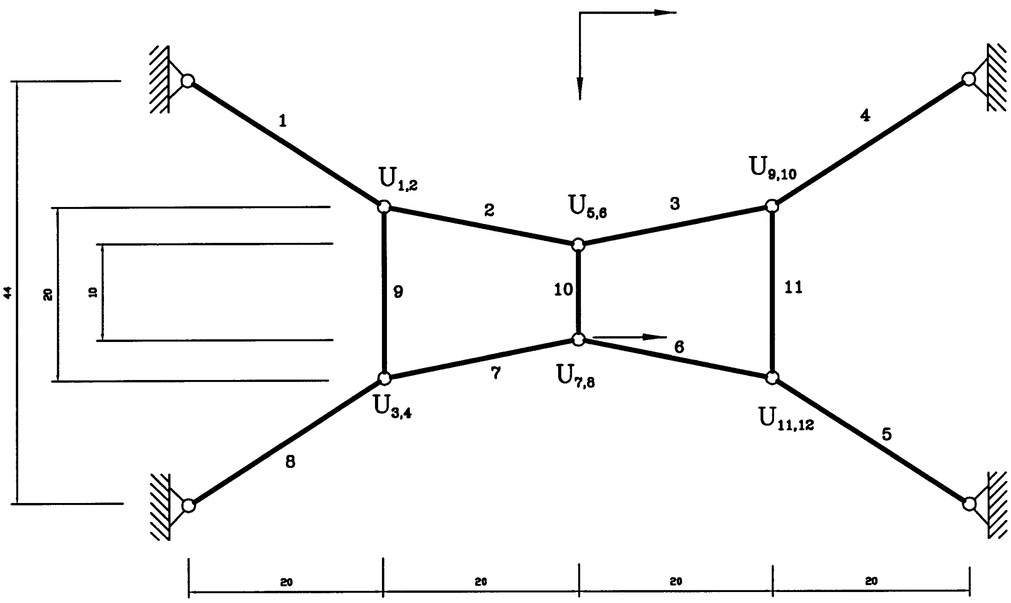

Fig. 1. Underconstrained plane net (all dimensions are in cm).

Table 1
Plane net analysis

	Prestressing forces $\mathbf{P}_0$ (kg)	Displacements (cm) ($\mathbf{P}_0$)	Force increments ($\mathbf{P}_0$)	Displacements (cm) (0.1 $\mathbf{P}_0$)	Force increments (0.1 $\mathbf{P}_0$)
1	33	−0.135761	23.1873	−0.17569	39.7813
2	29.1682	0.227331	19.7734	0.294745	34.4443
3	29.1682	0.1372	19.6517	0.177651	34.3223
4	33	0.226088	21.3767	0.29262	37.9665
5	33	−0.0829065	12.7875	−0.108687	29.4107
6	29.1682	0.0176042	10.6307	0.0303911	25.3461
7	29.1682	0.0859907	29.0548	0.113316	43.7643
8	33	0.0165455	31.9703	0.0285656	48.5926
9	9.90404	−0.15017	8.17951	−0.200579	13.1367
10	14.1486	−0.250158	9.69981	−0.334102	16.8598
11	9.90404	0.152086	5.0151	0.203302	9.96021
12	—	−0.252062	—	−0.33746	—

Table 2
Space assembly analysis

	Prestressing forces $\mathbf{P}_0$ (kg)	Displacements (cm) ($\mathbf{Q}$)	Force increments ($\mathbf{Q}$)	Displacements (cm) ($2\mathbf{Q}$)	Force increments ($2\mathbf{Q}$)
1	30	0	38.9322	0	75.8429
2	30.1851	0.0875694	38.8896	0.125641	75.8608
3	34.0683	−0.152945	42.8128	−0.216299	83.4779
4	30	0	24.1757	0	47.6218
5	30.1851	0.0526575	24.5373	0.0735608	48.206
6	34.0683	0.0898643	28.7056	0.147574	56.3406
7	30.1851	0	2.22877	0	4.8447
8	32.7342	0	2.41305	0	5.22822
9	30.1851	0.584405	0.137208	0.825119	1.04135
10	32.7342	0	0.151596	0	1.15318
11	31.1655	−0.0819677	2.60681	−0.114789	6.36395
12	31.1655	−0.152817	9.19334	−0.216098	17.4541
13	30	0	2.18023	0	4.77063
14	30	−0.0452143	0.158709	−0.0599611	1.05587
15	30	0.0895431	2.22353	0.146988	5.67294
16	30.1851	0	2. 28674	0	5.77998
17	30	0	9.17411	0	17.3467
18	30.1851	0.585078	9.18752	0.826467	17.3998
19	11.9404	0.0118521	22.2768	0.0160398	44.4486
20	7.46271	0.0141493	1.71253	0.0232076	3.44285
21	17.9107	−0.0310265	13.5703	−0.0507007	26.4951
22	2.985	−0.0078483	0.114383	−0.0130434	0.281827
23	10.4478	0.0078483	0.633409	0.0130434	1.44567
24	8.95536	0.011755	1.60787	0.0186052	3.20851
25	—	−0.0011755	—	−0.0161192	—
26	—	−0.0139707	—	−0.0227718	—
27	—	−0.0310046	—	−0.0506517	—
28	—	0.0076036	—	0.012449	—
29	—	−0.0076036	—	−0.012449	—
30	—	0.0117472	—	0.0185816	—
31	—	0.0321671	—	0.0456988	—
32	—	−0.0321671	—	−0.0456988	—
33	—	0.106818	—	0.151813	—
34	—	−0.0324732	—	−0.0463135	—
35	—	0.0324732	—	0.0463135	—
36	—	0.10684	—	0.151851	—

nodal displacements and force increments where external force of 18 kg is applied at the central bottom node (see figures). In the case of the plane net this load is horizontal and in the case of the space assembly it is vertical. The fourth and fifth columns of Table 1 present nodal displacements and force increments of the net where pre-stressing forces are ten times smaller (load is the same).

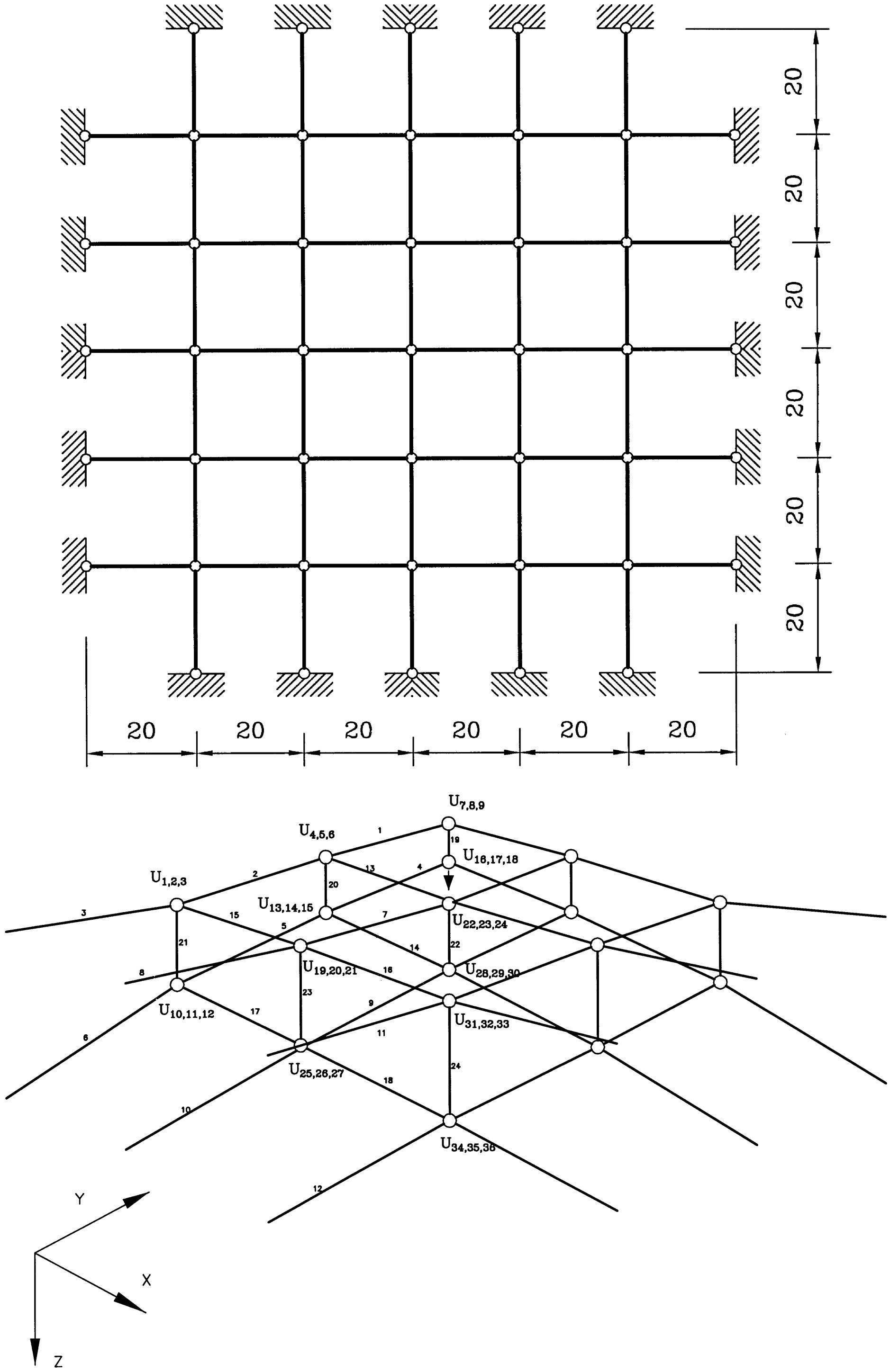

Fig. 2. Underconstrained space assembly: bottom view and repeated quarter (vertical distances between supporting points: 40; length of vertical members: $l_{19} = 8$, $l_{20} = l_{22} = 12$, $l_{21} = 18$, $l_{23} = 22$, $l_{24} = 28$; all dimensions are in cm).

Table 3
Convergence rates for plane net analysis

Scheme	Prestressing: $\mathbf{P}_0$	Prestressing: 0.1 $\mathbf{P}_0$
DNR (FDNR) exact kinematics	5 (5)	11 (9)
DNR (FDNR) small strains	5 (5)	11 (9)
DNR (FDNR) small displacements	5 (5)	11 (9)
SDNR (SFDNR) small strains	6 (6)	12 (9)

Table 4
Convergence rates for space assembly analysis

Scheme	Load: $\mathbf{Q}$	Load: 2$\mathbf{Q}$
DNR (FDNR) exact kinematics	6 (5)	8 (6)
DNR (FDNR) small strains	6 (5)	8 (6)
DNR (FDNR) small displacements	6 (5)	8 (6)
SDNR (SFDNR) small strains	9 (10)	14 (12)

The fourth and the fifth columns of Table 2 present nodal displacements and force increments of the space assembly where the load is doubled (pre-stressing forces are the same).

Tables 3 and 4 present convergence (number of iterations) of various computational schemes with tolerance $\beta = 0.005$ for all four cases of loading. Table 3 is related to the plane net and Table 4 to the space assembly. The second row of both tables presents convergence of displacement and force–displacement Newton–Raphson schemes based on exact kinematics. The DNR scheme based on exact kinematics is the most popular approach and may be found in most advanced texts on structural analysis or nonlinear finite element analysis. The FDNR scheme with exact kinematics was used by Szabo and Kollar (1984). The third and fourth rows present convergence of DNR and FDNR schemes based on small strains and displacement assumptions correspondingly. It is evident from the obtained results that the convergence rate is invariant with respect to kinematics adopted. Consequently, kinematics based on the assumption of small strains is preferable as the simplest one.

The fifth row of the tables presents convergence of subspace DNR and FDNR schemes. Their convergence slightly slows down in comparison to previous schemes, however, dimension of the inverted matrix is reduced from 12 to 2 in case of the net and from 150 to 15 in case of the space assembly.

6. Concluding remarks

Displacement and force–displacement formulations of the Newton–Raphson scheme with their "subspace modifications" were considered for analyses of underconstrained structures based on various kinematic assumptions: "exact kinematics", "small strains", "small displacements". Obtained results suggest the following conclusions:

- Choice of kinematic equations does not influence convergence rate and, consequently, the assumption of small strains is preferable computationally;
- Both displacement and force–displacement formulations of the Newton–Raphson technique provide approximately the same rate of convergence;
- Subspace modifications of the classical Newton–Raphson schemes allow to reduce significantly dimensions of inverted matrices without sensitive reduction of the convergence rate;
- Displacements of underconstrained structures are really small: exact analysis leads to the same results as analysis based on small displacement assumptions.

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