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Plane frames as semi-underconstrained systems

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Abstract

A different approach to the analysis of slender framed structures is presented. It is based on the theory of underconstrained systems. It is shown that frames can be interpreted as semi-underconstrained structures. The approach allows understanding of various features of the mechanical behavior of frames as well as novel computing techniques. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The assembly shown in Fig. 1a is a *finite mechanism* which, generally, does not resist external loads. There are various ways in which it is possible to transform the mechanism into a structure which can bear external loads. This transformation is obtained, for example, by imposing additional *constraints* on the degrees of freedom of the assembly. A possible way is adding the diagonal element shown in Fig. 1b, this new constraint introduces rigidity to the initial assembly. In this case, the price of the rigidity is rather high: the assembly weight grows up and its internal space is obstruct by the new element. These two features are undesirable in structural design.

It is possible also to apply initial forces to the assembly in such a way that it becomes stiff. Fig. 1c exhibits this possibility. The two initial vertical forces insert rigidity to the assembly. It becomes an *infinitesimal mechanism* or *underconstrained* structure: it possesses infinitesimal free motions only, and the overall stiffness is reached due to appropriate initial member forces. This example is

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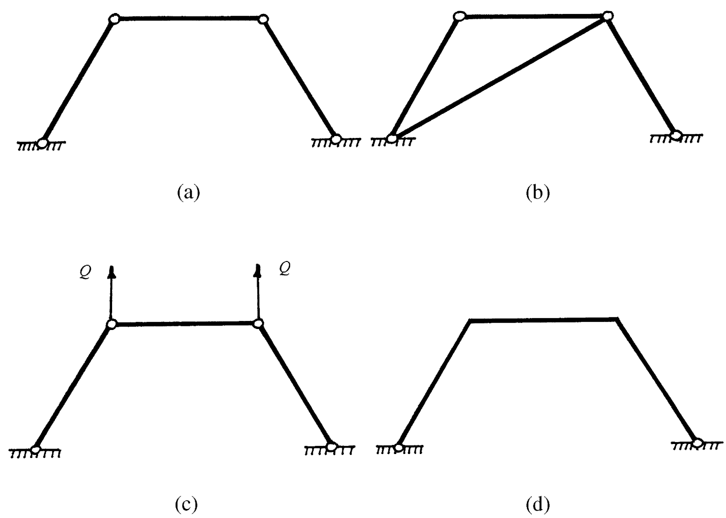


Fig. 1. Mechanism, truss, underconstrained structure, frame.

a simple illustration of an underconstrained system; it is somehow artificial since the initial vertical forces are hardly realizable practically. Generally, the initial stress state and rigidity may be obtained due to prestressing [1–4]. A more conventional way of introducing rigidity to the assembly is shown in Fig. 1d. In this case the pin joints are replaced by stiff connections and the assembly is transformed into a framed structure. By comparing the structures shown in Figs. 1c and d, it is possible to conclude that the role of the stiffening of the joints in the case of frames (Fig. 1d) is analogous to the role of the initial stress in the case of underconstrained structures (Fig. 1c). Because of the resemblance between underconstrained structures and frames, there is a certain similarity in their behavior. In this work the theory and analysis methods developed to underconstrained structures [5] are extended and adopted to frames.

2. Fundamentals

2.1. Equilibrium of frames

Equilibrium of the nodes of a framed structure is provided where generalized nodal displacements $\mathbf{v}$ satisfy the following equation:

$$\mathbf{Kv} = \mathbf{r}, \tag{1}$$

where $\mathbf{K}$ is the $m \times m$ stiffness matrix and $\mathbf{r}$ is the vector of the external loads. Generally, $\mathbf{K}$ possesses the following structure:

$$\mathbf{K} = \mathbf{B}^{\mathrm{T}}\mathbf{SB}, \tag{2}$$

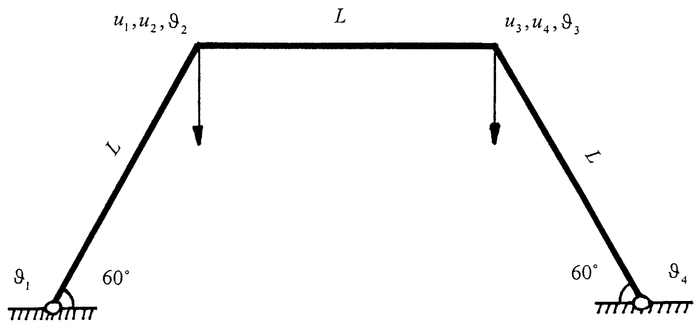


Fig. 2. A frame.

where **B** is an $n \times m$ kinematic matrix and **S** is an $n \times n$ uncoupled stiffness matrix. The generalized axial forces and bending moments **p** take the form

$$\mathbf{p} = \mathbf{SBu}. \tag{3}$$

In the case of the frame shown in Fig. 2, which consists of identical elements with cross-section area A, elasticity modulus E, length L, and moment of inertia I, Eq. (1) takes the form (see also appendix):

$$\frac{EA}{L}\left[\begin{array}{cccc|cccc}
1.3+9\varepsilon & -0.4+5.2\varepsilon & -1 & 0 & -519.6\varepsilon & -519.6\varepsilon & 0 & 0\\
 & 0.8+15\varepsilon & 0 & -12\varepsilon & -300\varepsilon & 300\varepsilon & 600\varepsilon & 0\\
 & & 1.3+9\varepsilon & 0.4-5.2\varepsilon & 0 & 0 & -519.6\varepsilon & -519.6\varepsilon\\
 & & & 0.8+15\varepsilon & 0 & -600\varepsilon & -300\varepsilon & 300\varepsilon\\
\hline
 & & & & 40000\varepsilon & 20000\varepsilon & 0 & 0\\
 & & & & & 80000\varepsilon & 20000\varepsilon & 0\\
 & \mathit{Symmetry} & & & & & 80000\varepsilon & 20000\varepsilon\\
 & & & & & & & 40000\varepsilon
\end{array}\right]$$

$$\times \begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ \vartheta_1\\ \vartheta_2\\ \vartheta_3\\ \vartheta_4 \end{bmatrix} = \mathbf{r}. \tag{4}$$

Dimensionless parameter $\varepsilon = I/(AL^2)$ is used above. In practical cases this parameter is very small. However, by observation of Eq. (4)[1], the entries including this parameter may not be small. This is due to the fact that displacements u_i and rotations ϑ_i have different dimensions, and because of it their contributions to the stiffness matrix cannot be compared directly.

In order to exclude rotations, matrix $\mathbf{K}$ can be partitioned. In this case Eq. (1) takes the form

$$\bar{\mathbf{K}}\mathbf{u} = \mathbf{q}, \tag{5}$$

where

$$\bar{\mathbf{K}} = \mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}, \tag{6}$$

$$\mathbf{q} = \mathbf{r}_1 - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{r}_2, \tag{7}$$

$\mathbf{u}$ is a vector of nodal displacements u_i, matrix $\mathbf{K}_{11}$ is submatrix of $\mathbf{K}$ corresponding to nodal displacements and external nodal forces, matrix $\mathbf{K}_{22}$ is submatrix of $\mathbf{K}$ corresponding to rotations and external nodal moments, matrices $\mathbf{K}_{12} = \mathbf{K}_{21}^{\mathrm{T}}$ are chosen appropriately. Vector $\mathbf{r}_1$ includes external nodal forces only while vector $\mathbf{r}_2$ includes external nodal moments only.

Solution of Eq. (5) immediately provides solution of Eq. (1):

$$\mathbf{v} = \begin{bmatrix} \mathbf{u} \\ \vartheta \end{bmatrix} = \begin{bmatrix} \mathbf{u} \\ -\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\mathbf{u} + \mathbf{K}_{22}^{-1}\mathbf{r}_2 \end{bmatrix}. \tag{8}$$

In the example, considered here, matrix $\mathbf{K}_{11}$ is the 4×4 upper left submatrix of Eq. (4), matrix $\mathbf{K}_{22}$ is the 4×4 lower right submatrix of Eq. (4), matrices $\mathbf{K}_{12}$ and $\mathbf{K}_{21}$ are the 4×4 upper right and lower left submatrices of Eq. (4) correspondingly. In this case the reduced stiffness matrix $\bar{\mathbf{K}}$ takes the form

$$\bar{\mathbf{K}} = \bar{\mathbf{K}}_{axial} + \bar{\mathbf{K}}_{bending}, \tag{9}$$

$$\bar{\mathbf{K}}_{axial} = \frac{EA}{L}\begin{bmatrix} 1.3 & -0.4 & -1 & 0 \\ & 0.8 & 0 & 0 \\ & & 1.3 & 0.4 \\ Symmetry & & & 0.8 \end{bmatrix},$$

$$\bar{\mathbf{K}}_{bending} = \varepsilon\frac{EA}{L}\begin{bmatrix} 1.2 & 2.4 & 0.3 & -1.9 \\ & 6.4 & 1.9 & -6.1 \\ & & 1.2 & -2.4 \\ Symmetry & & & 6.4 \end{bmatrix}.$$

Matrix $\bar{\mathbf{K}}$ is a sum of two matrices; the first matrix is related to the axial stiffness of the assembly while the second one is related to the bending stiffness of the assembly. The elements of the first matrix are dominant, they are much larger than the elements of the second one. Nevertheless, the second matrix cannot be ignored since the first matrix is singular. The fact that the first matrix is

[1] Here and further numerical results are significantly rounded for the sake of compactness.

dominant and singular is crucial to the analysis of the structure. This feature of frames makes them similar to underconstrained structures. This fact can be easily demonstrated by studying the assembly shown in Fig. 1c.

2.2. Equilibrium of underconstrained systems

The analogy of the stiffness matrices of frames and underconstrained structures is studied explicitly considering the example studied before. The frame shown in Fig. 2 can be transformed into an underconstrained structure where the stiff joints are replaced by pin joints and initial stress state is introduced due to vertical nodal forces Q as shown in Fig. 1c. In this case the stiffness matrix of the underconstrained structure takes the form (see [5] for details)

$$\mathbf{K} = \mathbf{K}_{axial} + \mathbf{K}_{prestress}, \tag{10}$$

$$\mathbf{K}_{axial} = \frac{EA}{L}\begin{bmatrix} 1.3 & -0.4 & -1 & 0 \\ & 0.8 & 0 & 0 \\ & & 1.3 & 0.4 \\ \mathit{Symmetry} & & & 0.8 \end{bmatrix},$$

$$\mathbf{K}_{prestress} = \mu\frac{EA}{L}\begin{bmatrix} 1.7 & 0 & -0.6 & 0 \\ & 1.7 & 0 & -0.6 \\ & & 1.7 & 0 \\ \mathit{Symmetry} & & & 1.7 \end{bmatrix},$$

in which μ is a parameter; $\mu = Q/(EA)$. μ is small because Q initial stresses are significantly smaller than the modulus of elasticity.

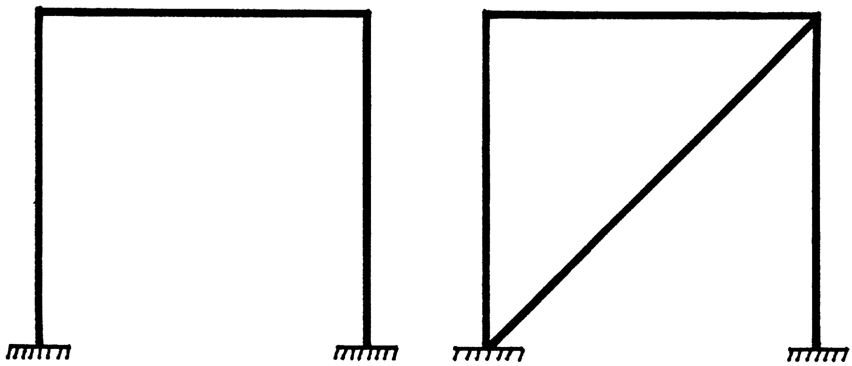
Eqs. (9) and (10) demonstrate the similarity of the stiffness matrices of frames and underconstrained structures. The first terms on the right-hand sides of Eqs. (9) and (10) are identical because these terms correspond to the axial stiffness of the bars which is the same in both cases. The second terms of Eqs. (9) and (10), being both small in comparison to the first ones, are different. In the case of frames they are related to the bending stiffness, and in the case of underconstrained structures, to the initial stresses. The similar nature of the stiffness matrices allows to extend the analysis method developed for underconstrained structures to frames.

It should be stressed that frames are not underconstrained systems, they are not infinitesimal mechanisms and do not possess infinitesimal motions, and, strictly speaking, cannot be called underconstrained. Thus, it is better to refer to frames as *semi-underconstrained.*

2.3. Decomposition of the stiffness matrix and its ill-conditionedness

The reduced stiffness matrix of frames may be represented in the form

$$\bar{\mathbf{K}} = \bar{\mathbf{K}}_{axial} + \bar{\mathbf{K}}_{bending}, \tag{11}$$



(a) (b)

Fig. 3. The portal frame.

in which

$$\|\bar{\mathbf{K}}_{axial}\| \gg \|\bar{\mathbf{K}}_{bending}\|, \tag{12}$$

where

$$Det(\bar{\mathbf{K}}_{axial}) = 0. \tag{13}$$

The fact that matrix $\bar{\mathbf{K}}_{axial}$ which corresponds to the axial stiffness of the elements is singular means that the *skeletal* (truss) structure, where stiff connections are replaced by pin joints, is a finite mechanism and the bending stiffness of the joints is necessary to provide the overall stiffness of the frame. If matrix $\bar{\mathbf{K}}_{axial}$ is of full rank, the second term on the right-hand side of Eq. (11) could be ignored and the assembly does not need any additional stiffness to the axial stiffness of its members. Such frames are actually trusses in which the stiffness of their joints is not necessary for their overall stiffness.

Mathematically, in the case of frames, matrix $\bar{\mathbf{K}}$ is ill-conditioned because of Eqs. (12) and (13).

The fact that the stiffness matrix of frames is ill-conditioned was pointed out already by Livesley [6], who studied the behavior of the simple portal frame shown in Fig. 3a. He also noted that the addition of the diagonal element (Fig. 3b) transforms the stiffness matrix from ill-conditioned to the well-conditioned. These observations are in good accordance with the analysis presented here. It should be noted that the use of the stiffness matrix $\mathbf{K}$ instead of the reduced stiffness matrix $\bar{\mathbf{K}}$ may lead to ill-conditionedness even for potentially well-conditioned problems of the type shown in Fig. 3b because of the different dimensions of rotations and displacements. Thus, generally, it is safer to use matrix $\bar{\mathbf{K}}$.

2.4. Orthogonal decomposition of displacements

The fact that $Det(\bar{\mathbf{K}}_{axial}) = 0$ implies the following decomposition of the displacements:

$$\mathbf{u} = \mathbf{u}_a + \mathbf{u}_b, \tag{14}$$

$$\mathbf{u}_a = \tilde{\mathbf{W}}\tilde{\mathbf{z}}, \quad \mathbf{u}_b = \mathbf{W}\mathbf{z}. \tag{15}$$

Vector $\mathbf{u}_a$ is associated with nodal displacements due to axial deformations of the members and it is a linear combination of vectors which form the orthonormal basis of the range of matrix $\bar{\mathbf{K}}_{axial}$. Vector $\mathbf{u}_b$ is associated with nodal displacements in which no axial deformations of the members take place and only bending of members occurs. It is represented by a linear combination of vectors which form the orthonormal basis of the null space of matrix $\bar{\mathbf{K}}_{axial}$. Consequently, the columns of matrix $\mathbf{W}$ span the null space of $\bar{\mathbf{K}}_{axial}$ and the columns of matrix $\tilde{\mathbf{W}}$ span the range of $\bar{\mathbf{K}}_{axial}$

$$\bar{\mathbf{K}}_{axial}\mathbf{W} = \mathbf{0}, \quad \mathbf{W}^{\mathrm{T}}\tilde{\mathbf{W}} = \mathbf{0}. \tag{16}$$

By using Eqs. (14) and (15), Eq. (5), pre-multiplied by matrix $[\tilde{\mathbf{W}}\mathbf{W}]^{\mathrm{T}}$ from the left, takes the form

$$\begin{bmatrix} \mathbf{K}_a & \mathbf{L} \\ \mathbf{L}^{\mathrm{T}} & \mathbf{K}_b \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{z}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{W}}^{\mathrm{T}} & \mathbf{q} \\ \mathbf{W}^{\mathrm{T}} & \mathbf{q} \end{bmatrix}, \tag{17}$$

where

$$\mathbf{K}_a = \tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{K}}\tilde{\mathbf{W}}, \quad \mathbf{K}_b = \mathbf{W}^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{W}, \quad \mathbf{L} = \tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{W}. \tag{18}$$

The new (reduced) stiffness matrix, transformed from the old one, possesses a more attractive framework. The left upper square submatrix $\mathbf{K}_a$, of the dimension equals to the $Rank(\bar{\mathbf{K}}_{axial})$, is dominant since its elements depend on the axial stiffness. The rest submatrices $\mathbf{K}_b$ and $\mathbf{L}$ depend on the bending stiffness only

$$\mathbf{K}_b = \mathbf{W}^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{W} = \mathbf{W}^{\mathrm{T}}\bar{\mathbf{K}}_{bending}\mathbf{W}, \quad \mathbf{L} = \tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{W} = \tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{K}}_{bending}\mathbf{W}. \tag{19}$$

In the example presented here, matrices $\mathbf{W}$, $\tilde{\mathbf{W}}$, $\mathbf{K}_a$, $\mathbf{K}_b$, and $\mathbf{L}$ take the form

$$\mathbf{W} = \begin{bmatrix} -0.61 \\ -0.35 \\ -0.61 \\ 0.35 \end{bmatrix}, \quad \tilde{\mathbf{W}} = \begin{bmatrix} -0.79 & 0 & 0 \\ 0.27 & -0.77 & 0.45 \\ 0.47 & 0.59 & 0.24 \\ -0.27 & 0.24 & 0.86 \end{bmatrix}, \tag{20}$$

$$\mathbf{K}_a = \frac{EA}{L}\begin{bmatrix} 2 & 0.32 & 0.55 \\ & 1.04 & 0.31 \\ \mathit{Symmetry} & & 0.96 \end{bmatrix} + \varepsilon\frac{EA}{L}\begin{bmatrix} 1.92 & -1.71 & -1.2 \\ & 4.47 & 1.47 \\ \mathit{Symmetry} & & 0.81 \end{bmatrix},$$

$$\mathbf{K}_a \cong \frac{EA}{L}\begin{bmatrix} 2 & 0.32 & 0.55 \\ & 1.04 & 0.31 \\ \mathit{Symmetry} & & 0.96 \end{bmatrix}, \tag{21}$$

$$\mathbf{K}_b = \varepsilon\frac{EA}{L}[8], \quad \mathbf{L} = \varepsilon\frac{EA}{L}\begin{bmatrix} -3.1 \\ 5.74 \\ 2.35 \end{bmatrix}.$$

In fact, the proposed congruence transformation leads to the concentration of the elements of large numerical value in the left upper submatrix. This domination leads to well-conditioned submatrices at the upper left and lower right. Indeed, by using Laplace's resolution of determinant, it is

Table 1

		Eigenvalues				Condition numbers: $\kappa_2(\bullet) = \|\bullet\|_2 \|\bullet^{-1}\|_2$
$\bar{\mathbf{K}}$	$\varepsilon = 10^{-5}$:	2.37	1.0	0.63	8×10^{-5}	~ 30000
$(EA/L = 1)$	$\varepsilon = 10^{-3}$:	2.37	1.0	0.63	8×10^{-3}	~ 300
$\bar{\mathbf{K}}_a$	$\varepsilon = 10^{-5}$:	2.37	1.0	0.63	—	~ 4
$(EA/L = 1)$	$\varepsilon = 10^{-3}$:	2.37	1.0	0.63	—	~ 4
$\bar{\mathbf{K}}_b$	$\varepsilon = 10^{-5}$:	—	—	—	8×10^{-5}	1
$(EA/L = 1)$	$\varepsilon = 10^{-3}$:	—	—	—	8×10^{-3}	1

possible to find, approximately, "large" eigenvalues of the reduced stiffness matrix $\bar{\mathbf{K}}$ as eigenvalues of matrix $\mathbf{K}_a$ and "small" eigenvalues of $\bar{\mathbf{K}}$ as eigenvalues of matrix $\mathbf{K}_b$. Thus, the spectrum of each submatrix is narrower than the spectrum of the stiffness matrix $\bar{\mathbf{K}}$ as a whole. These submatrices are significantly better conditioned as compared to the stiffness matrix. Table 1 represents eigenvalues and the 2-norm condition numbers κ_2 of matrices $\bar{\mathbf{K}}$, $\mathbf{K}_a$, $\mathbf{K}_b$ for various magnitudes of ε considering the example discussed here.

Since the submatrices of Eq. (17) are well-conditioned, the block Gaussian elimination procedure is expected to be accurate considering round-off error.

2.5. Fitted and non-fitted loads

It is useful to rewrite Eq. (17) in the following form:

$$(\mathbf{K}_a - \mathbf{L}\mathbf{K}_b^{-1}\mathbf{L}^{\mathrm{T}})\tilde{\mathbf{z}} = (\tilde{\mathbf{W}}^{\mathrm{T}} - \mathbf{L}\mathbf{K}_b^{-1}\mathbf{W}^{\mathrm{T}})\mathbf{q}, \tag{22}$$

$$(\mathbf{K}_b - \mathbf{L}^{\mathrm{T}}\mathbf{K}_a^{-1}\mathbf{L})\mathbf{z} = (\mathbf{W}^{\mathrm{T}} - \mathbf{L}^{\mathrm{T}}\mathbf{K}_a^{-1}\tilde{\mathbf{W}}^{\mathrm{T}})\mathbf{q}. \tag{23}$$

The second terms in the parentheses on the left-hand sides of Eqs. (22) and (23) are small as compared to the first ones. Particularly, the second terms take the following forms for the considered numerical example:

$$\mathbf{L}\mathbf{K}_b^{-1}\mathbf{L}^{\mathrm{T}} = \varepsilon\frac{EA}{L}\begin{bmatrix} 1.2 & -2.22 & -0.91 \\ & 4.12 & 1.69 \\ Sym. & & 0.69 \end{bmatrix},$$

$$\mathbf{L}^{\mathrm{T}}\mathbf{K}_a^{-1}\mathbf{L} = \varepsilon^2\frac{EA}{L}[48.4].$$

These terms may be ignored and Eqs. (22) and (23) take the form:

$$\tilde{\mathbf{z}} = \mathbf{K}_a^{-1}(\tilde{\mathbf{W}}^{\mathrm{T}} - \mathbf{L}\mathbf{K}_b^{-1}\mathbf{W}^{\mathrm{T}})\mathbf{q}, \tag{24}$$

$$\mathbf{z} = \mathbf{K}_b^{-1}(\mathbf{W}^{\mathrm{T}} - \mathbf{L}^{\mathrm{T}}\mathbf{K}_a^{-1}\tilde{\mathbf{W}}^{\mathrm{T}})\mathbf{q}. \tag{25}$$

This formulation allows to distinguish between "axial" and "bending" displacements $\mathbf{u}_a(\tilde{\mathbf{z}})$ and $\mathbf{u}_b(\mathbf{z})$.

(i) If the external load $\mathbf{q}$ belongs to the range of matrix $\bar{\mathbf{K}}_{axial}$, that is $\mathbf{W}^{\mathrm{T}}\mathbf{q} = \mathbf{0}$, then

$$\tilde{\mathbf{z}} = \mathbf{K}_a^{-1}\tilde{\mathbf{W}}^{\mathrm{T}}\mathbf{q}, \tag{26}$$

$$\mathbf{z} = -\mathbf{K}_b^{-1}\mathbf{L}^{\mathrm{T}}\mathbf{K}_a^{-1}\tilde{\mathbf{W}}^{\mathrm{T}}\mathbf{q} \tag{27}$$

and "bending" displacements are not larger than "axial" ones: $\|\mathbf{u}_b\| \sim \|\mathbf{u}_a\|$. This kind of loading may be called *non-exciting* or *fitted* since the skeletal structure (with nodal pins) bears the given load and the contribution of the bending stiffness is negligible.

(ii) If the external load is general, i.e. it possesses both fitted and *non-fitted* component which belongs to the null space of matrix $\bar{\mathbf{K}}_{axial}$ ($\mathbf{W}^{\mathrm{T}}\mathbf{q} \neq \mathbf{0}$), then

$$\tilde{\mathbf{z}} = \mathbf{K}_a^{-1}(\tilde{\mathbf{W}}^{\mathrm{T}} - \mathbf{L}\mathbf{K}_b^{-1}\mathbf{W}^{\mathrm{T}})\mathbf{q}, \tag{28}$$

$$\mathbf{z} \cong \mathbf{K}_b^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{q} \tag{29}$$

and "bending" displacements are significantly larger than "axial" ones: $\|\mathbf{u}_b\| \gg \|\mathbf{u}_a\|$ and the contribution of the bending stiffness in bearing given loads is crucial.

In the case of the frame shown in Fig. 2, where the cross-section of the elements is 1×1 cm, length $L = 100$ cm, and the elasticity modulus $E = 2 \times 10^6$ kg/cm^2, the following "bending" and "axial" displacements for fitted and non-fitted loading are obtained:

$$\mathbf{q}_{fitted} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}(\mathrm{kg}), \quad \mathbf{u}_a = \begin{bmatrix} 1.4 \\ 7.5 \\ -1.4 \\ 7.5 \end{bmatrix} \times 10^{-5}\,(\mathrm{cm}), \quad \mathbf{u}_b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}(\mathrm{cm}), \tag{30}$$

$$\mathbf{q}_{non\text{-}fitted} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1/2 \end{bmatrix}(\mathrm{kg}), \quad \mathbf{u}_a = \begin{bmatrix} 0.8 \\ 6.1 \\ -1.3 \\ 5.2 \end{bmatrix} \times 10^{-5}\,(\mathrm{cm}), \quad \mathbf{u}_b \begin{bmatrix} 8.1 \\ 4.7 \\ 8.1 \\ -4.7 \end{bmatrix} \times 10^{-2}\,(\mathrm{cm}). \tag{31}$$

The fact that the "bending" displacements are dominant in the case of non-fitted load indicates that geometrically non-linear analysis is necessary as in the case of underconstrained systems [7]. The appropriate numerical example is considered in the next section.

3. Buckling

Overall buckling of statically loaded elastic structures is usually associated with drastic and visible changes in their configuration. The problem of buckling analysis may be formulated as a non-linear eigenvalue problem. Its solution requires tracing of the equilibrium paths in the state space and checking their stability.

Traditionally, structural analysis is restricted to the classical linear buckling analysis formulated by a linear eigenvalue problem (see, for example, [8]). Despite of its relative simplicity the formal assumption of linear buckling may lead to incorrect conclusions in cases where behavior of the structure is non-linear.

Table 2

Critical loads\\t	1	4	8
Exact method	92.3751	23644.5	378128
Degenerate method	92.3768	23651.3	378566

Table 3

	u_1	u_2	u_3	u_4	ϑ_1	ϑ_2	ϑ_3	ϑ_4
Exact	0.612	0.353	0.612	− 0.353	0.012	− 0.002	− 0.002	0.012
Degenerate	0.612	0.353	0.612	− 0.353	0.012	− 0.002	− 0.002	0.012

Below the classical linear buckling analysis of frames is studied by using the approach developed in this work, and it is shown how the validity of the classical linear buckling can be examined.

3.1. Classical linear buckling analysis

The linear buckling analysis is associated with the (generalized) eigenvalue problem

$$(\mathbf{K} + v\mathbf{D})\mathbf{v} = \mathbf{0}, \tag{32}$$

where $\mathbf{D}$ is the *geometrical* stiffness matrix; v is a load parameter: $\mathbf{r} = v\mathbf{r}^*$ ($\mathbf{r}^*$ is a fixed initial load vector); and vector $\mathbf{v}$ represents buckling modes. Matrix $\mathbf{D}$ is associated with internal bending moments and axial forces computed from Eqs. (1)–(3) with external load $\mathbf{r}^*$ (see appendix for details). Generally, only the smallest eigenvalue $v_{\min}$ and the corresponding eigenvector are required and the critical buckling load takes the form: $\mathbf{r}_{critical} = v_{min}\mathbf{r}^*$.

Let the frame considered in this work be loaded as follows: $\mathbf{r}^* = [0, 1, 0, 1, 0, 0, 0, 0]^{\mathrm{T}}$ (kg). The vertical nodal forces are in the directions of u_2, u_4 (see Fig. 2). The critical buckling loads for various square cross-section ($t \times t$) were found and are given in the first row of Table 2. The buckling mode corresponding to $\mathbf{r}_{critical}$ for $t = 1$ cm is shown in the first row of Table 3.

In the case where the approach considered in Section 2 is applied to Eq. (32), the reduced linear buckling equation takes the form

$$(\bar{\mathbf{K}} + v\bar{\mathbf{D}})\mathbf{u} = \mathbf{0}. \tag{33}$$

The reduced stiffness matrix $\bar{\mathbf{K}}$ is defined by using Eq. (6) and the reduced geometrical stiffness matrix $\bar{\mathbf{D}}$ is obtained by removing zero rows and columns corresponding to rotational degrees of freedom from matrix $\mathbf{D}$.

The problem may be reduced further with the help of the congruence transformation described in Section 2.4:

$$\begin{bmatrix} \mathbf{K}_a + v\tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{W}} & \mathbf{L} + v\tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W} \\ \mathbf{L}^{\mathrm{T}} + v\mathbf{W}^{\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{W}} & \mathbf{K}_b + v\mathbf{W}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{z}} \\ \mathbf{z} \end{bmatrix} = \mathbf{0} \tag{34}$$

and eigenvalues may be found from the following equation:

$$\Psi(v) = det\left\{\begin{bmatrix} \mathbf{K}_a + v\tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{W}} & \mathbf{L} + v\tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W} \\ \mathbf{L}^{\mathrm{T}} + v\mathbf{W}^{\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{W}} & \mathbf{K}_b + v\mathbf{W}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W} \end{bmatrix}\right\} = \mathbf{0}. \tag{35}$$

Since $\|\mathbf{K}_a\| \gg \|\mathbf{L}\|, \|\bar{\mathbf{D}}\|, \|\mathbf{K}_b\|$, then the left upper submatrix is dominant under condition that $v \ll \|\mathbf{K}_a\|$. The latter is satisfied for "small" eigenvalues which are of the most interest. Particularly, the lowest eigenvalue, corresponding to the critical load, is of interest in the buckling problem. Proceeding further with Laplace's resolution of determinant, it is possible to obtain approximately

$$\begin{aligned}\Psi(v) &= det\{(\mathbf{K}_a + v\tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{W}})(\mathbf{K}_b + v\mathbf{W}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W}) + \text{small terms}\} \\ &\cong det\{(\mathbf{K}_a + v\tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{W}})(\mathbf{K}_b + v\mathbf{W}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W})\} \\ &= det\{(\mathbf{K}_a + v\tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{W}})\}det\{(\mathbf{K}_b + v\mathbf{W}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W})\} = 0.\end{aligned} \tag{36}$$

Recalling the fact that $\|\mathbf{K}_a\| \gg \|\mathbf{K}_b\|$, Eq. (36) is further reduced to

$$det(\mathbf{K}_b + v\mathbf{W}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W}) = 0. \tag{37}$$

This equation could be naturally called the reduced buckling equation, or, keeping in mind that reduction was made twice, it is better to call it the *degenerate linear buckling equation*. It may be directly observed from this degenerate equation that the critical buckling load is affected by the bending stiffness of the frame elements ($\mathbf{K}_b$).

The buckling mode correspondent to $v_{\min}$, computed by using Eq. (34), takes the form: $\mathbf{u} = \mathbf{W}\mathbf{z} + \tilde{\mathbf{W}}\tilde{\mathbf{z}}$, where $\mathbf{z}$ lies in the null space of matrix $(\mathbf{K}_b + v_{\min}\mathbf{W}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W})$ and $\tilde{\mathbf{z}} = (\mathbf{K}_a + v_{\min}\tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{W}})^{-1}(\mathbf{L} + v_{\min}\tilde{\mathbf{W}}^{\mathrm{T}}\bar{\mathbf{D}}\mathbf{W})\mathbf{z}$. However, $\|\tilde{\mathbf{z}}\| \ll \|\mathbf{z}\|$ and the buckling mode takes the following form:

$$\mathbf{u} \cong \mathbf{W}\mathbf{z}. \tag{38}$$

The buckling mode as a function of $\mathbf{v}$ is obtained by using Eqs. (8) and (38).

Tables 2 and 3 show the results obtained by using the exact method and the degenerate equations presented in this work. Comparison of the exact and degenerate methods does not need any comment. It should be noted that the reduction of the dimensions of the problem is significant. The exact method eigenvalue problem is an 8×8 while in the degenerate method it is 1×1 only.

3.2. Applicability of the linear buckling analysis

Linear buckling analysis is valid in cases where the external loads are fitted loads

$$\mathbf{W}^{\mathrm{T}}\mathbf{q} = \mathbf{0}. \tag{39}$$

Sufficiency of this requirement is obvious. In this case, the external loads are equilibrated by the axial forces only, the bending stiffness is not necessary and the skeletal truss bears the loads.

There are two arguments to support the *necessity* of this statement.

The first one is "by analogy", it was shown in Section 2.5 that in the case of fitted loads the "bending" displacements are of the same order of magnitude as the "axial" ones, thus non-linear terms of equilibrium equations can be ignored, and the case is of linear buckling. Only in the case of non-fitted loads the "bending" displacements are dominant and non-linear terms of equilibrium equations should be retained. This case is analogous to the case of non-fitted loads of underconstrained systems. Non-linear analysis is necessary when "kinematic" (analogous to "bending") displacements are significantly larger than "elastic" (analogous to "axial") ones (see [7]). For the numerical example considering here, it is possible to find that linear buckling analysis leads to the value $v_{\min} = 123$ in the case of non-fitted loading represented in Section 2.5. If this value were valid, both linear and nonlinear analyses up to the corresponding critical load would give the same displacements. Computations show significant differences in numerical results of linear and non-linear analyses even at $v = 70$

$$\mathbf{r} = [0, 70, 0, 35, 0, 0, 0, 0]^{\mathrm{T}} \text{ (kg, kg cm)},$$

$$\mathbf{v}_{linear} = [5.68, 3.29, 5.68, -3.28, 5.47, -2.19, -2.19, 5.47]^{\mathrm{T}} \text{ (cm, rad)},$$

$$\mathbf{v}_{non\text{-}linear} = [13.36, 9.24, 12.19, -5.97, 13.29, -4.4, -6.01, 11.69]^{\mathrm{T}} \text{ (cm, rad)}.$$

Non-linear computations were carried out on the base of the co-rotational 2D-beam element described in appendix.

The second argument is supported by the following property of the bifurcation point [9];

$$\mathbf{q}^{\mathrm{T}}\mathbf{u} = \mathbf{0}, \tag{40}$$

in which vector **u** represents an appropriate buckling mode. It was shown in Section 3.1 that this buckling mode may be represented by Eq. (38), and one returns to Eq. (38) after substitution of Eq. (38) in Eq. (40). Thus the critical point is a bifurcation point in the case of fitted loads only. Otherwise, this critical point is a limit point. The limit point, however, cannot be reached, in principle, within the frame of linear analysis.

4. Vibrations

The equations of undamped free vibrations take the form

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{0}, \tag{41}$$

where **M** is a "lumped" mass matrix with diagonal elements or a "consistent" mass matrix including non-diagonal elements [10]. The latter is of special interest where the finite element formulation is used. In this case matrix **M** should be consistent with the base functions approximation. In the case where exact beam elements are used the consistency of the "consistent" mass matrix is less obvious. Because of its simplicity, the diagonal mass matrix is preferable. In practical cases, independently of matrix **M** structure

$$\|\mathbf{K}\| \gg \|\mathbf{M}\|. \tag{42}$$

The amplitude–frequency transformation of Eq. (41) takes the following form:

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{v} = \mathbf{0}, \tag{43}$$

where ω is an angular frequency and $\mathbf{v}$ is the corresponding vector of free vibrations amplitudes. By excluding rotations from Eq. (43) it takes the form

$$(\bar{\mathbf{K}} - \omega^2 \bar{\mathbf{M}})\mathbf{u} = \mathbf{0}, \tag{44}$$

where $\bar{\mathbf{K}}$, $\bar{\mathbf{M}}$ are reduced stiffness and mass matrices ($\|\bar{\mathbf{K}}\| \gg \|\bar{\mathbf{M}}\|$).

Eq. (44) represents a generalized eigenvalue problem of free vibrations and it may be treated in the same manner as the linear buckling eigenvalue problem considered before in Section 3.1, it is only necessary to replace matrix $\bar{\mathbf{D}}$ of the buckling problem by matrix $\bar{\mathbf{M}}$ and parameter v by parameter $-\omega^2$. It is possible to write the *degenerate linear vibrations equation*

$$det\{(\mathbf{K}_b - \omega^2 \mathbf{W}^{\mathrm{T}} \bar{\mathbf{M}} \mathbf{W})\} = 0 \tag{45}$$

and the amplitudes of vibration modes take the form

$$\mathbf{u} \cong \mathbf{W}\mathbf{z}. \tag{46}$$

Approximations (45) and (46) are valid for the most important frequencies — the low frequencies. The number of columns of matrix $\mathbf{W}$ defines their number. It may be observed again, like in the case of the degenerate buckling Eq. (37), that the low frequencies of free vibration are affected by the bending stiffness of the frame members.

Assuming that the mass matrix of the numerical example discussed in this work (Fig. 2) consists of unit diagonal elements corresponding to nodal translations only, Eq. (45) degenerates to 1×1 case and the squared low angular frequency coincides with the eigenvalue of matrix $\mathbf{K}_b$ shown in the bottom row of Table 1, the whole spectrum of squared angular frequencies is given in the second row of the same table.

5. Concluding remarks

A specific approach to the analysis of frames has been developed in this paper. Frames are considered as underconstrained trusses whose lack of constraints is compensated by the bending stiffnesses of the joints. In this case there are two sources of rigidity to the frame: the bending and axial stiffness of the elements, in which the bending stiffness is small to compare to the axial one. The latter affects the whole structural behavior. This fact implies that:

- linear equilibrium equations in terms of displacements are ill-conditioned;
- because of the ill-conditionedness the geometrically non-linear analysis may be necessary for the correct solution;
- the low frequencies of free vibrations as well as the critical buckling (bifurcation) load are affected by the bending stiffness and not the axial stiffness of frames.

All the qualitative conclusions, considered above, were obtained by considering the reduced equilibrium equations in terms of displacements in combination with a specific orthogonal decomposition of displacements. This method leads to some novel computational schemes which make computations more reliable and simple. In particular:

- A simplified (degenerate) forms of linear buckling and vibrations analysis are proposed (Eqs. (37) and (45)). These forms allow to significantly reduce the problems' dimensions.
- A simple sufficient criterion of the validity of linear buckling analysis is introduced (Eq. (39)).
- A block form of equilibrium equations is presented (Eq. (17)) allowing improvement of the problem conditioning and, consequently, avoiding accumulation of rounding errors.

Appendix. 2D beam

The co-rotational 2D beam element [11] and its linearization is used for computations within the work. The sample kth element between the ith and jth nodes is shown before and after deformation in Fig. 4. Kinematic relations take the form

$$\Delta_k = l_k - L_k, \quad \vartheta_{ki} = \vartheta_i - \alpha_k, \quad \vartheta_{kj} = \vartheta_j - \alpha_k, \tag{A.1}$$

where

$$l_k = \sqrt{(x_j + u_j - x_i - u_i)^2 + (x_{j+1} + u_{j+1} - x_{i+1} - u_{i+1})^2},$$

$$L_k = \sqrt{(x_j - x_i)^2 + (x_{j+1} - x_{i+1})^2},$$

$$\alpha_k = Arc\sin\{[(x_j - x_i)(u_{j+1} - u_{i+1}) - (x_{j+1} - x_{i+1})(u_j - u_i)](l_k L_k)^{-1}\}, \quad -\pi/2 \leqslant \alpha_k \leqslant \pi/2.$$

x_i, x_{i+1} (x_j, x_{j+1}) are horizontal and vertical coordinates of the corresponding nodes and u_i, u_{i+1} (u_j, u_{j+1}) are appropriate displacements.

Constitutive relations take the form

$$\mathbf{p}_k = \mathbf{S}_k \mathbf{e}_k \quad \text{or} \quad \begin{bmatrix} N_k \\ M_{ki} \\ M_{kj} \end{bmatrix} = \frac{E_k A_k}{L_k} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4r_k^2 & 2r_k^2 \\ 0 & 2r_k^2 & 4r_k^2 \end{bmatrix} \begin{bmatrix} \Delta_k \\ \vartheta_{ki} \\ \vartheta_{kj} \end{bmatrix}, \tag{A.2}$$

where N_k, M_{ki}, M_{kj} are axial force and bending momdents of the kth member, $r_k = \sqrt{I_k/A_k}$ is the radius of gyration, and E_k is the elasticity modulus.

By differentiating Eq. (A.1) the transposed equilibrium matrix of the kth member $\mathbf{B}_k$ is obtained

$$\delta \mathbf{e}_k = \mathbf{B}_k \delta \mathbf{v}_k,$$

$$\begin{bmatrix} \delta\Delta_k \\ \delta\vartheta_{ki} \\ \delta\vartheta_{kj} \end{bmatrix} = \begin{bmatrix} -c_k & -s_k & c_k & s_k & 0 & 0 \\ -s_k/l_k & c_k/l_k & s_k/l_k & -c_k/l_k & 1 & 0 \\ -s_k/l_k & c_k/l_k & s_k/l_k & -c_k/l_k & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta u_i \\ \delta u_{i+1} \\ \delta u_j \\ \delta u_{j+1} \\ \delta\vartheta_i \\ \delta\vartheta_j \end{bmatrix}, \tag{A.3}$$

in which

$$c_k = (x_j + u_j - x_i - u_i)/l_k, \quad s_k = (x_{j+1} + u_{j+1} - x_{i+1} - u_{i+1})/l_k.$$

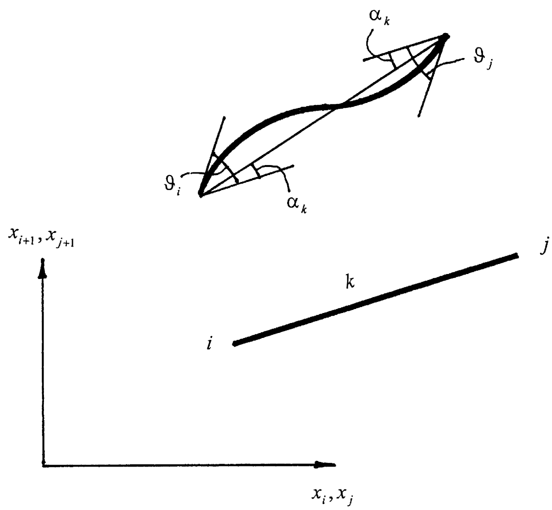


Fig. 4. The co-rotational 2D beam element.

The kth element tangent stiffness matrix $\mathbf{C}_k$ takes the form

$$\mathbf{C}_k = \mathbf{B}_k^{\mathrm{T}} \mathbf{S}_k \mathbf{B}_k + \mathbf{D}_k. \tag{A.4}$$

The geometrical stiffness matrix $\mathbf{D}_k$ is present as follows:

$$\mathbf{D}_k = \frac{\partial(\mathbf{B}_k^{\mathrm{T}} \mathbf{p}_k)}{\partial \mathbf{v}_k}, \tag{A.5}$$

where $\mathbf{p}_k$ is fixed. Componentwisely, Eq. (A.5) takes the form

$$\mathbf{D}_k = \left[\begin{array}{cccc|cc} -2csM + s^2N & -csN + (c^2 - s^2)M & 2csM - s^2N & csN - (c^2 - s^2)M & 0 & 0 \\ & c^2N + 2csM & csN - (c^2 - s^2)M & -c^2N - 2csM & 0 & 0 \\ & & -2csM + s^2N & -csN + (c^2 - s^2)M & 0 & 0 \\ & & & c^2N + 2csM & 0 & 0 \\ \hline & \mathit{Symmetry} & & & 0 & 0 \\ & & & & & 0 \end{array}\right],$$

$$c \equiv c_k, \quad s \equiv s_k, \quad N \equiv N_k l_k^{-1}, \quad M \equiv (M_{ki} + M_{kj}) l_k^{-2}.$$

Gathering tangent stiffness matrices of all members and imposing boundary conditions it is possible to obtain the global tangent stiffness matrix for non-linear analysis

$$\mathbf{C} = \mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{B} + \mathbf{D}. \tag{A.6}$$

The usual stiffness matrix $\mathbf{K}$ for linear analysis is obtained from $\mathbf{C}$ by zeroing all initial displacements, axial forces and bending moments (Eq. (2) of the paper)

$$\mathbf{K} = \mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{B}|_{\mathbf{v}=\mathbf{0}}. \tag{A.7}$$

The classical linear buckling analysis is performed by adding matrix $\mathbf{D}$ to matrix $\mathbf{K}$ where axial forces and bending moments are obtained from the linear analysis for some initial fixed load $\bar{\mathbf{r}}$ (Eq. (32) of the paper).

References

[1] Calladine CR. Buckminster Fuller's tensegrity structures and Clerk Maxwell's rules for the construction of stiff frames. International Journal of Solids and Structures 1978;14:161–72.
[2] Kuznetsov EN. Underconstrained Structural Systems. New York: Springer, 1991.
[3] Pellegrino S. A class of tensegrity domes. International Journal of Space Structures 1992;7:127–42.
[4] Volokh K Yu, Vilnay O. New classes of reticulated underconstrained structures. International Journal of Solids and Structures 1997;34:1093–104.
[5] Volokh K Yu, Vilnay O. "Natural", "kinematic" and "elastic" displacements of underconstrained structures. International Journal of Solids and Structures 1997b;34:911–30.
[6] Livesley RK. Matrix methods of structural analysis (2nd ed.). Oxford: Pergamon Press, 1975.
[7] Volokh K Yu. Non-linear analysis of underconstrained structures. International Journal of Solids and Structures 1999;36:2175–87.
[8] Timoshenko SP, Gere JM. Theory of elastic stability (2nd ed.). New York: McGraw-Hill, 1961.
[9] Riks E. Buckling analysis of elastic structures: a computational approach. Advances in Applied Mechanics 1998;34:1–76.
[10] Argyris J, Mlejnek H-P. Dynamics of structures. Amsterdam: North-Holland, 1991.
[11] Crisfield MA. Non-linear finite element analysis of solids and structures. New York: Wiley, 1991.