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On the prediction of geometrical non-linearity of slender structures

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SUMMARY

A criterion of the validity of geometrically linear structural analysis is proposed. It is based on the results of the linear analysis and the knowledge of the appropriate non-linear strain measures. Geometrically non-linear computations are avoided. A plane cable net and a space star-shaped dome are used for the demonstration of the relevance of the criterion. Copyright © 2001 John Wiley & Sons, Ltd.

1. INTRODUCTION

Slender structures like shells, frames, cable nets, etc., are the main bearing elements of modern engineering systems. Because of their slenderness, sophisticated techniques of geometrically non-linear methods of analysis are required. There are, at least, two important points underlying the necessity of non-linear analysis.

First the non-linear analysis leads to a more accurate determination of the internal forces affecting the structural strength. The belief that the linear analysis leads to larger (in absolute values) internal forces than those actually occur and hence the errors are contained in the safety factor is not always correct. The real forces in the structure may be significantly larger than those obtained by using linear analysis.

Second, it is often (tacitly) assumed that the structure deforms linearly till buckling occurs. This leads to the linear eigenvalue problem for buckling analysis. However, the assumption of the linearity up to the critical load is not correct in many practical cases. Thus, the use of the linear buckling analysis is questionable and non-linear analysis of buckling is necessary.

Practically, the non-linear analysis became accessible with the enormous growth of computer power and the development of non-linear numerical methods in recent years. Thus enthusiasts of computer methods suggest replacing the linear analysis by non-linear one. In principle, that solves the problem. Unfortunately, the non-linear analysis is still non-trivial. It requires a high level of numerical expertise and experience, and so it is still a 'computational art' to a large extent, and it is wise to restrict the non-linear analysis to the cases where it is unavoidable. So it is important to predict the necessity of the non-linear analysis. This problem is rarely

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discussed in the literature. The general advice is to check the smallness of the displacement gradients as compared to unity in the case of shells or to check the smallness of the ratios of nodal displacements to the proper member lengths in the case of framed structures. The check is performed after linear analysis is carried out. Though the idea seems to be reasonable, the checking parameter is uncertain. For example, let the gradient entry is of the order of 10^{-2}: is it small? Is 10^{-1} or 10^{-3} small enough? A distinct and reliable answer could hardly be given. This was the reason why a different criterion was established in the theory of plates and shells. In those structures it is recommended to check the ratio between linear lateral displacements and the shell thickness. If the ratio is less than 1/3 or 1/4 then the linear theory is supposed to be valid. This criterion was derived from the analytical solutions of some simple problems and its general validity is still questionable.

Some qualitative criterion of the linear behaviour of framed structures with pin and rigid joints may be extracted from the authors' papers [1–3]. This criterion considers a skeleton of the real structure. In the case of trusses, cable nets, tensegrity systems the skeleton is the structure itself; in the case of frames pins replace rigid joints. Every skeleton, even though it is a mechanism, possesses its own *equilibrium* or *fitted* loads consistent with the initial equilibrium equations; and the skeleton resists the fitted loads in its undeformed configuration. The criterion implies that framed structures behave linearly if the loads are fitted. The criterion is qualitative in its nature; it requires neither linear nor non-linear analysis. However, this criterion does not allow an exact prediction of the critical level of the load in which the linear structural behaviour is practically separated from the non-linear one.

In the present work a simple numerical criterion which allows an exact prediction of the stage in which non-linear analysis is necessary is proposed. This criterion requires linear analysis and examines the appropriate non-linear strain measures without carrying any non-linear analysis. Numerical examples of a plane cable net and a space star-dome are used for testing the proposed criterion and comparison with the prediction drawn from the qualitative criterion described above. However, it should be clearly realized that the proposed criterion is correct for regular structural behaviour and cannot predict the non-linearity that appears at bifurcation points, where branching occurs, or their sharp unfoldings in the presence of small imperfections. An in-length discussion of this important issue appears in the Conclusions.

2. THE CRITERION: CONTINUUM FORMULATION

Let the general static boundary value problem (BVP) for slender elastic bodies be formulated as follows:

- *Equilibrium*:

$$\nabla \cdot \{(\mathbf{S} + \mathbf{S}_0) \cdot (\mathbf{1} + \nabla \mathbf{u})\} = \mathbf{0}, \quad \nabla \cdot \mathbf{S}_0 = \mathbf{0} \tag{1}$$

- *Material*:

$$\mathbf{S} = \mathbf{C} : \mathbf{E} \tag{2}$$

- *Kinematics*:

$$\mathbf{E} = \left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}\right)/2 + \nabla \mathbf{u} \cdot \nabla \mathbf{u}^{\mathrm{T}}/2 \tag{3}$$

- *Boundary conditions*:

$$\{(\mathbf{S}+\mathbf{S}_0)\cdot(\mathbf{1}+\nabla\mathbf{u})\}\cdot\mathbf{N}=\mathbf{t},\quad \mathbf{S_0}\cdot\mathbf{N}=\mathbf{0}\quad \text{on } \partial\Omega^t$$
$$\mathbf{u}=\mathbf{0}\quad \text{on } \partial\Omega^u \tag{4}$$

Here Lagrangian description with reference to the initial body configuration Ω is used: ∇ is the Hamilton operator relatively the initial configuration, $\mathbf{S}$ and $\mathbf{S}_0$ are the second (symmetric) Piola–Kirchhoff tensors of stress increments and initial stresses correspondingly, $\mathbf{E}$ is the Green–Lagrange strain tensor, $\mathbf{1}$ is the unit tensor, $\mathbf{C}$ is the fourth-order elasticity tensor, $\mathbf{u}$ is a displacement vector, $\mathbf{N}$ is a vector of unit outward normal to the body surface $\partial\Omega$ and $\mathbf{t}$ is a vector of the surface load per the initial surface area.

In the above formulation volume forces are ignored, the surface loads are 'dead' and strains are small: $\|\mathbf{E}\| \ll \|\mathbf{1}\|$. The latter allows Hooke's law (Equation (2)). Deformations of the body, however, may be significant and the problem is generally non-linear. Such assumptions cover a wide range of slender structures without refering to a specific configuration.

In the case where deformations are small:

$$\|\nabla\mathbf{u}\| \ll \|\mathbf{1}\| \quad \text{and} \quad \mathbf{E} \cong (\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}})/2 \tag{5}$$

the non-linear BVP may be approximately replaced by the appropriate linear BVP:

- *Equilibrium*:

$$\nabla\cdot(\mathbf{S}+\mathbf{S}_0\cdot\nabla\mathbf{v})=\mathbf{0} \tag{6}$$

- *Material*:

$$\mathbf{S}=\mathbf{C}:\mathbf{E} \tag{7}$$

- *Kinematics*:

$$\mathbf{E}=\left(\nabla\mathbf{v}+\nabla\mathbf{v}^{\mathrm{T}}\right)/2 \tag{8}$$

- *Boundary conditions*:

$$(\mathbf{S}+\mathbf{S}_0\cdot\nabla\mathbf{v})\cdot\mathbf{N}=\mathbf{t},\quad \mathbf{S}_0\cdot\mathbf{N}=\mathbf{0}\quad \text{on } \delta\Omega^t$$
$$\mathbf{v}=\mathbf{0}\quad \text{on } \delta\Omega^v \tag{9}$$

Vector $\mathbf{v}$ is used here instead of $\mathbf{u}$ in order to underline the difference between linear and non-linear formulations.

Let the load vector field be changed proportionally to the load parameter λ:

$$\mathbf{t}=\lambda\mathbf{q} \tag{10}$$

Then the linear displacements are proportional to the load parameter while the non-linear ones are not:

$$\mathbf{v}=\mathbf{v}[\lambda\mathbf{q}]=\lambda\mathbf{v}[\mathbf{q}] \tag{11}$$

When the linear problem is well posed there is a range, at least infinitesimal, of λ where solutions of linear and non-linear BVPs are close. With the growth of λ, these solutions generally diverge. The latter may be found by using an iterative procedure in which the linear solution $\mathbf{v}$ is substituted into the non-linear equations (1)–(4). In this case a non-trivial residual is obtained. This residual may be used to correct the linear solution at the next step of the iterative procedure of solving non-linear problems.

In this work a different approach to the problem is considered. The non-linear problem is not solved; instead, the value of the load parameter λ_{cr}, which indicates the upper boundary up to which linear analysis is valid, is formulated:

$$\mathbf{u}[\lambda_{cr}\mathbf{q}] \cong \mathbf{v}[\lambda_{cr}\mathbf{q}] \tag{12}$$

Substituting linear solution $\mathbf{v}$ into Equations (1)–(4), it is possible to obtain a non-trivial residual or its specific measure as a function of λ. λ_{cr}, which is the boundary between correct and non-correct results of the linear analysis, may be obtained by minimizing or bounding the residual function. Similar approach is known in non-linear analysis as the 'linear search' technique.

This way to obtain λ_{cr} requires the performance of the first stage of the general non-linear analysis since the general non-linear equations are used. A significant simplification is achieved where λ_{cr} is obtained by considering small deformations in the following form:

$$\max_{\Omega}\left\{\frac{\|\nabla\mathbf{u}\cdot\nabla\mathbf{u}^{\mathrm{T}}/2\|}{\|(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}})/2\|}\right\} \leqslant \varepsilon \tag{13}$$

where ε indicates the error of linear strains and stresses as compared to the non-linear ones. This magnitude will be extensively discussed further. Taking into account that in the vicinity of the critical value of λ, separating linear and non-linear structural behaviour: $\mathbf{u}[\lambda_{cr}\mathbf{q}] = \mathbf{v}[\lambda_{cr}\mathbf{q}] = \lambda_{cr}\mathbf{v}[\mathbf{q}]$, it is possible to obtain instead of (13):

$$\lambda_{cr} = \varepsilon \times \min_{\Omega}\left\{\frac{\|\left(\nabla\mathbf{v}[\mathbf{q}]+\nabla\mathbf{v}[\mathbf{q}]^{\mathrm{T}}\right)/2\|}{\|\left(\nabla\mathbf{v}[\mathbf{q}]\cdot\nabla\mathbf{v}[\mathbf{q}]^{\mathrm{T}}\right)/2\|}\right\} \tag{14}$$

This criterion allows to obtain the critical load factor separating geometrical linearity and non-linearity. The tolerance used in the criterion is equally applicable to both strains and stresses because of Hooke's linear constitutive equations. Thus, the accuracy of strains and stresses is restricted to the same magnitude. Unfortunately, analytical solutions of the formulated linear BVP are rarely available and the criterion should be reformulated for the discretized problem.

3. THE CRITERION: DISCRETE FORMULATION

The discrete BVP approximating the exact continuous equations is ordinarily obtained with the help of finite element techniques. These techniques may be applied directly to Equations (1)–(4) or (6)–(9), or they are applied after simplification of the equations using assumptions of the so-called 'engineering theories' accounting for specific geometrical configurations of slender bodies [4]. Such applications are, generally, accompanied by the 'weak form' variational reformulation of the BVP. In any case, the general structure of discrete equations remains similar:

- *Equilibrium*:

$$\int \mathbf{B}^{\mathrm{T}}(\mathbf{p}_0+\mathbf{p})\,\mathrm{d}\Omega = \mathbf{q}, \quad \int \mathbf{B}_0^{\mathrm{T}}\mathbf{p}_0\,\mathrm{d}\Omega = \mathbf{0} \tag{15}$$

- *Material*:

$$\mathbf{p} = \mathbf{C}\mathbf{e} \tag{16}$$

- *Kinematics*:

$$\mathbf{e} = \mathbf{e}_0[\mathbf{u}] + \mathbf{e}_1[\mathbf{u}, \mathbf{u}] \tag{17}$$

The boldface upper case letters are used henceforth for square and rectangular matrices and boldface lower case letters are used for column matrices, $\mathbf{p}, \mathbf{p}_0$ and $\mathbf{e}$ are column matrices of member stress increments, initial stresses and strains correspondingly, $\mathbf{q}$ and $\mathbf{u}$ are column matrices of nodal external forces and displacements, $\mathbf{C}$ is an elastic stiffness matrix, $\mathbf{B}^{\mathrm{T}}$ is an equilibrium matrix for the final configuration, $\mathbf{B}_0^{\mathrm{T}}$ is an equilibrium matrix for the initial configuration, column matrices $\mathbf{e}_0$ and $\mathbf{e}_1$ are linear and bilinear forms of $\mathbf{u}$ correspondingly, and the usual element assembling procedure with integration over the volume is applied.

The relationship between equilibrium and kinematics may be traced from the virtual work principle and it takes the form:

$$\delta\mathbf{e} = \mathbf{B}\delta\mathbf{u} \tag{18}$$

$$\mathbf{B} = \frac{\partial \mathbf{e}_0}{\partial \mathbf{u}} + \frac{\partial \mathbf{e}_1}{\partial \mathbf{u}} \equiv \mathbf{B}_0 + \mathbf{B}_1[\mathbf{u}] \tag{19}$$

The discrete linear BVP is obtained as follows:

- *Equilibrium*:

$$\int \left(\mathbf{B}_0^{\mathrm{T}}\mathbf{p} + \mathbf{D}\mathbf{v}\right) \,\mathrm{d}\Omega = \mathbf{q}, \quad \mathbf{D} = \frac{\partial\left(\mathbf{B}^{\mathrm{T}}\mathbf{p}_0\right)}{\partial \mathbf{u}} \tag{20}$$

- *Material*:

$$\mathbf{p} = \mathbf{C}\mathbf{e} \tag{21}$$

- *Kinematics*:

$$\mathbf{e} = \mathbf{e}_0[\mathbf{v}] = \mathbf{B}_0\mathbf{v} \tag{22}$$

λ_{cr} is obtained from Equation (14) which takes the form

$$\lambda_{\mathrm{cr}} = \varepsilon \times \min_j \frac{\|(\mathbf{e}_0[\mathbf{v}])_j\|}{\|(\mathbf{e}_1[\mathbf{v}, \mathbf{v}])_j\|} \tag{23}$$

where $(\bullet)_j$ means the jth entry of the corresponding matrix.

4. NUMERICAL EXAMPLES

Three structures, shown (in cm) in Figures 1–3 are considered to illustrate the previous reasoning and to choose the tolerance ε. All structures comprise straight members connected by pins at nodes. In this case the discretization is naturally obtained:

$$l_j^2(\mathbf{e}_0)_j = (x_i - x_s)(u_i - u_s) + (x_{i+1} - x_{s+1})(u_{i+1} - u_{s+1}) + (x_{i+2} - x_{s+2})(u_{i+2} - u_{s+2}) \tag{24}$$

$$2l_j^2(\mathbf{e}_1)_j = (u_i - u_s)^2 + (u_{i+1} - u_{s+1})^2 + (u_{i+2} - u_{s+2})^2 \tag{25}$$

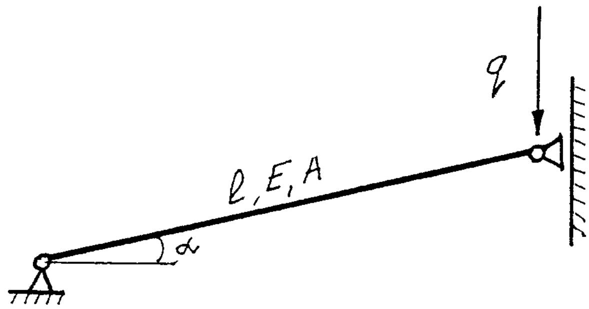

Figure 1.

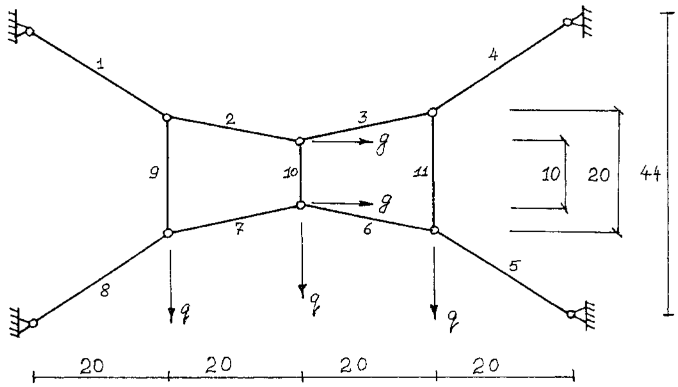

Figure 2.

where l_j is the jth member initial length and x_i, u_i are the appropriate nodal co-ordinates and displacements.

4.1. A one-degree-of-freedom system

The one-degree-of-freedom system shown in Figure 1 is examined in order to demonstrate the principle of the proposed criterion. In this case, the linear and bilinear strain measures take the following simple forms:

$$\mathbf{e}_0 = \{-u \sin\alpha/l\}, \quad \mathbf{e}_1 = \{u^2/(2l^2)\} \tag{26}$$

Substituting these values into Equations (17), (16), (19), (15) successively, it is possible to obtain the following non-linear equilibrium equation in terms of displacements:

$$(\sin\alpha)^2 u - \frac{3\sin\alpha}{2l}u^2 + \frac{1}{2l^2}u^3 = \lambda\frac{ql}{EA} \tag{27}$$

where λ is the loading parameter, q is a fixed trial load, E is the elasticity modulus and A is the cross-sectional area.

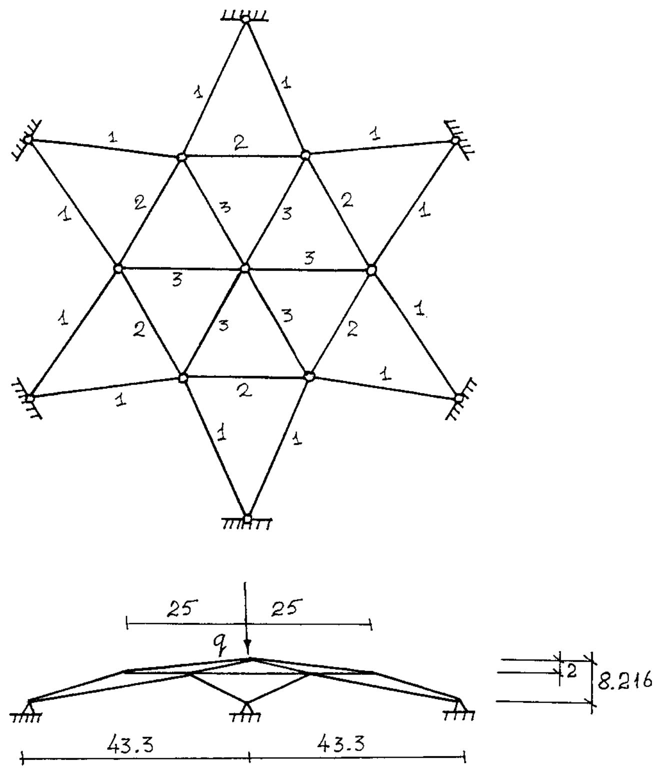

Figure 3.

The linear solution v is obtained when the second and the third terms on the left-hand side of Equation (27) are ignored and $\lambda = 1$:

$$v = \frac{ql}{(\sin\alpha)^2 EA} \tag{28}$$

Criterion (23) for the critical load parameter λ separating linear and non-linear behaviours is obtained from Equations (26) and (28):

$$\lambda_{\mathrm{cr}} = \varepsilon \frac{2l^2 v \sin\alpha}{lv^2} = \varepsilon \frac{2lEA(\sin\alpha)^3}{ql} \tag{29}$$

Let now $\lambda = \lambda_{\mathrm{cr}}$ and $u = \lambda_{\mathrm{cr}} v$ be substituted into the equilibrium equation (27) with account of Equations (23) and (24). It is obtained in this case:

$$\varepsilon - 3\varepsilon^2 + 2\varepsilon^3 = \varepsilon \tag{30}$$

In principle, this equation may be solved exactly. Such a solution, however, is of minor interest, since it does not cover different practical cases of slender structures. This equation, nonetheless, possesses the typical structure including the second- and the third-order terms with asymptotically equal coefficients. The second and the third order of ε can be ignored for

$\varepsilon \leqslant 0.01$. The latter justifies that the tolerance parameter should be of the order: $\varepsilon = 0.01$. A larger parameter may lead to the significant error in Equation (30).

4.2. Plane cable net

In the case of the cable net (Figure 2), the cross-sectional area and the modulus of elasticity are $A = 0.04\pi\,\mathrm{cm}^2$ and $E = 2.1 \times 10^6\,\mathrm{kg/cm}^2$ correspondingly for all members. The net is prestressed by the given forces:

$$\mathbf{p}_0 = \{33, 29.1682, 29.1682, 33, 33, 29.1682, 29.1682, 33, 9.90404, 14.1486, 9.90404\}^{\mathrm{T}}\,\mathrm{kg}$$

Two loading cases are considered. The first one is equal vertical nodal forces q and the second one is equal horizontal nodal forces g as shown in the figure. In the first case the load does not belong to the range of the initial equilibrium matrix $\mathbf{B}_0^{\mathrm{T}}$ thus it is a *non-fitted* load. The second loading case belongs to the range of the initial equilibrium matrix $\mathbf{B}_0^{\mathrm{T}}$ and thus it is a *fitted* load. By using the criterion described above, two critical expressions separating the linear and non-linear responses are obtained:

$$\begin{aligned} q_{\mathrm{cr}} &= 3.3\varepsilon\,\mathrm{kg} \\ g_{\mathrm{cr}} &= 5.8 \times 10^5 \varepsilon\,\mathrm{kg} \end{aligned} \tag{31}$$

Where λ_{cr} is predicted, it is necessary to check all the members of the structure because the structural members with largest (in absolute value) linear strains do not generally define the critical load. In this structure, for example, members 2 and 3 (Figure 2) define q_{cr} in Equation (31) while the most strained members are members 5 and 8.

In the case of the fitted horizontal loading, linear behaviour is observed for $\varepsilon = 0.01$ up to the value $g = 5.8\,\mathrm{t}$. The forces in some cables become zeros and the cables switch off at the load $g = 56\,\mathrm{kg}$. This result was verified by the non-linear analysis using Newton–Raphson technique. To this end the qualitative prediction of linearity is in good agreement with actual results. The quantitative criterion was also verified.

In the case of the non-fitted vertical loading the force increments for various loads, obtained from the linear and non-linear analyses, are shown in Table I. The quantitative criterion given by Equation (31) predicts that linear analysis can be used only up to $q = 0.033\,\mathrm{kg}$ where $\varepsilon = 0.01$. Non-linear analysis is required for loads exceeding 0.333 kg because then the non-linear and linear paths of the structure behaviour diverge. From Table I it can be seen that the error of linear analysis is about 10 per cent for $q = 0.33\,\mathrm{kg}$ (the second column of the table) and about 40 per cent for $q = 3.3\,\mathrm{kg}$ (the third column of the table) and there is nothing in common between the linear and non-linear analyses for $q = 33\,\mathrm{kg}$ (the fourth column of the table). Though the results for the loads $q = 0.33\,\mathrm{kg}$ and $q = 3.3\,\mathrm{kg}$ which correlate to the tolerances of $\varepsilon = 0.1$ and 1 do not exceed 10 and 100 per cent errors, respectively, it is necessary to be very careful with this fact. The good correspondence between these 'predictions' and final errors is more occasional than regular.

Two points are to be stressed in the case of the non-fitted loading. Firstly, the linear analysis leads to a significant underestimation of the forces (absolute values) in some cables. Secondly, in spite of the fact that the largest nodal displacement obtained form the linear analysis is 0.0617 cm for $q = 3.3\,\mathrm{kg}$ (0.617 cm for $q = 33\,\mathrm{kg}$) only, which is very small compared to the span of 20 cm, non-linear effects are significant.

Table I. Force increments of the cable net.

	$q = 0.033$ kg Linear non-linear	$q = 0.33$ kg Linear non-linear	$q = 3.3$ kg Linear non-linear	$q = 33$ kg Linear non-linear
1	0.045183 0.0454221	0.45183 0.473818	4.5183 6.478	45.183 87.5987
2	0.038861 0.0390255	0.38861 0.407852	3.8861 5.60417	38.861 75.6812
3	0.038861 0.0390255	0.38861 0.407852	3.8861 5.60417	38.861 75.6812
4	0.045183 0.0454221	0.45183 0.473818	4.5183 6.478	45.183 87.5987
5	−0.0484784 −0.0482394	−0.484784 −0.462809	−4.84784 −2.88907	−48.4784 −6.03866
6	−0.0417726 −0.0416073	−0.417726 −0.398415	−4.17726 −2.45218	−41.7726 −4.53774
7	−0.0417726 −0.0416073	−0.417726 −0.398415	−4.17726 −2.45218	−41.7726 −4.53774
8	−0.0484784 −0.0482394	−0.484784 −0.462809	−4.84784 −2.88907	−48.4784 −6.03866
9	0.0160035 0.016018	0.160035 0.166966	1.60035 2.21957	16.0035 13.1875
10	0.0157938 0.0159248	0.157938 0.166921	1.57938 2.37456	15.7938 31.9171
11	0.0160035 0.016018	0.160035 0.166966	1.60035 2.21957	16.0035 13.1875

4.3. Star-shaped dome

In the case of the star-shaped truss dome (Figure 3), the cross-sectional area and the modulus of elasticity are $A = 3.17\,\mathrm{cm}^2$ and $E = 3.03 \times 10^5\,\mathrm{kg/cm}^2$, correspondingly, for all members. Vertical load q is applied at the central node as shown in the figure. The critical load separating linear and non-linear behaviours obtained by using the proposed criterion is

$$q = 1890.81\varepsilon\,\mathrm{kg} \tag{32}$$

and in the case where $\varepsilon = 0.01$: $q = 18.9\,\mathrm{kg}$.

Table II presents the force increments obtained by using linear and non-linear analyses for $q = 18.9\,\mathrm{kg}$ and larger loads. As predicted the error of using linear analysis is about 1 per cent at $q = 18.9\,\mathrm{kg}$ and it is significantly larger for larger loads. It is important to realize that the error corresponding to the $\varepsilon = 0.1$, which is given by the second column of the table, is larger than 20 per cent. This result indicates that it is impossible to predict the accuracy of the linear analysis with 10 per cent error in strains or internal forces, only 1 per cent accuracy ($\varepsilon = 0.01$) seems to be the maximum valid accuracy as described in Section 4.1. The linear

Table II. Force increments of the truss dome.

	$q = 18.9\,\text{kg}$ Linear non-linear	$q = 189\,\text{kg}$ Linear non-linear	$q = 250\,\text{kg}$ Linear non-linear
1	-8.00682	-80.0682	-105.910
	-8.00552	-79.877	-105.495
2	30.1007	300.1007	396.958
	30.5641	368.906	550.294
3	-39.5008	-395.008	-522.497
	-39.9633	-462.818	-674.471

analysis diverges from the non-linear one at an early stage of loading and the force increments of the non-linear analysis are significantly larger (in absolute values) than those of the linear one.

The considered example imposes restrictions on the qualitative criterion described in Section 1. Indeed, the skeleton of the structure coincides with the structure itself. It resists any loading in its undeformed configuration. This means that any load is a fitted load and linear analysis should be valid. In contrast to this prediction the real behaviour of the structure becomes non-linear at very moderate loads. Such failure of the qualitative criterion may be explained by the fact that the given load leads to significant changes in equilibrium configuration and the structure tends to the snap through behaviour: the top members lie approximately in the same plane. From the mathematical point of view, in this case the columns of the equilibrium matrix become closer under loading since the top structural members become flatter. At some stage of loading the structural configuration becomes singular and the fitted load is transformed to the non-fitted one.

Finally, the importance of non-linear analysis should be stressed with respect to the buckling phenomenon. The formal linear buckling analysis, based on the assumption of the linear equilibrium path of the structure in its state space, gives the bifurcation point at the load: $q = 1520\,\text{kg}$. At the same time the accurate non-linear analysis gives the limit point (snap through) at the load: $q = 300\,\text{kg}$. Thus, there is the five times quantitative difference in the buckling load value. Besides, there is the qualitative difference in the character of the singular point: the linear analysis predicts the bifurcation point and the non-linear analysis predicts the limit point. The latter cannot be obtained in principle within the linear analysis framework.

5. CONCLUSIONS

A simple quantitative criterion (Equation (23)) allowing the prediction of the maximum load up to which linear analysis can be used was introduced. This maximum load is predicted by using the results of linear analysis and the evaluation of non-linear strains. This criterion is associated with the 1 per cent error of the strains and the stresses (internal forces) obtained by using linear analysis as compared to those found by using non-linear analysis. It was shown that it is impossible to predict the validity of the linear analysis with errors of the internal forces larger than 1 per cent.

The numerical examples presented in the work verify the proposed criterion and stress the necessity of geometrically non-linear analysis. It is shown that linear analysis only may lead to wrong and lower internal forces as well as to misunderstanding of the buckling phenomenon.

Finally, it is important to appreciate the limitations of the proposed criterion in the case of bifurcation points. It is evident, that the non-linearity that occurs at branching (bifurcation) points cannot be identified by the proposed simple criterion. This criterion misses secondary equilibrium paths departing from the bifurcation points. Nonetheless, this criterion is able to predict the *validity* of the *linear* bifurcation analysis, based on the assumption that the equilibrium path is linear up to the first singular point. That is, if the critical load parameter, separating linearity from non-linearity, is larger than the critical parameter of the bifurcation analysis, then the latter is correct and the bifurcation point lies on the linear equilibrium path. The situation, where the equilibrium path is supposed to be linear up to the first singular point, is generally assumed in the traditional linear buckling analysis; it may be wrong, however, as one can see from the example of the star-shaped dome shown (Section 4.3). A subtler situation may occur in the presence of *very small* imperfections leading to sharp unfoldings of the bifurcation point. In this case it is possible to imagine imperfections that do not allow the 'slow' development of the non-linearity: the equilibrium path is almost linear up to the bifurcation point and it 'suddenly switches' to the secondary branch. Such a situation of the 'sharp non-linearity' (without real branching) needs further investigations and exceeds the scope of this paper.

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