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# On foundations of the Hardy Cross method

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**Abstract**

It is shown that the Hardy Cross method for analysis of frames is Jacobi iterative scheme applied to the displacement formulation of structural analysis. Because of it convergence for any loading conditions is due to the strict diagonal dominance of the stiffness matrix.


*Keywords:* Moment distribution; Hardy Cross; Iterations; Jacobi scheme

## 1. Introduction

The Cross method, or the so-called moment distribution method, was originally proposed by Hardy Cross for analysis of framed structures in 1932 (Cross, 1932). This method is taught extensively at colleges (Utku et al., 1991; Tartaglione, 1991; Arbabi, 1991; Rossow, 1996; Ghali and Neville, 1997; Kassimali, 1999) and it still remains an important tool for analysis of simple frames in practice and can be programmed for complicated problems without the use of modern structural analysis software.

At the first stage of the analysis it is assumed that the rigid joints of frame members are initially fixed against rotations as in the displacement method. The reactive moments produced by external loads are computed. These moments are unbalanced at the joints of the original non-restrained structure. In order to equilibrate the joints the moments are distributed proportionally to the corresponding member stiffness. These distributed moments are associated with the so-called "carryover" moments at the opposite ends of structural members. They are considered to be new incremental unbalanced moments and the procedure repeats until the unbalanced moments become negligible. The true moments at the ends of all members are the sum of all distributed moment increments. This procedure assumes the joint rotations only. In the case where joint translations are significant a more general scheme is used, which requires applying the method successively.

The Cross method converges in all known examples, but the general proof of convergence was not established. Southwell (1940) considered the moment-distribution method in lines of his relaxation approach.

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Though various reasonable physical arguments (based on equilibrium and energy considerations) were proposed, the exhaustive formal proof of the convergence was not reached. Another attempt to tie the Cross method to a numerical iterative procedure can be found in Behravesh and Kaveh (1990).

In this note it is shown that the Cross method is nothing but Jacobi iterative scheme in disguise. This is done by revealing "displacement nature" of the Cross method. Particularly, it is shown that the method is the incremental form of Jacobi iterative scheme applied to the classical displacement formulation of the problem. The convergence of Jacobi iterations, and thus of the Cross method, is provided due to the strict diagonal dominance of the stiffness matrix.

## 2. Displacement method

The canonical system of equilibrium equations in terms of displacements takes the form:

$$\mathbf{K}\mathbf{u} + \mathbf{p} = 0, \tag{1}$$

where $\mathbf{K}$ is an $m$ by $m$ stiffness matrix; $\mathbf{u}$ is a vector of nodal displacements and $\mathbf{p}$ is a vector of nodal loads.

In the case where nodal translations are of secondary importance the displacement vector $\mathbf{u}$ comprises nodal rotations only: $u_j = \vartheta_j$. Entry $p_i$ of vector $\mathbf{p}$ is obtained as a sum of the end moments of structural members meeting at the $i$th joint. These moments are computed for joints clamped against rotations. Entry $K_{ij}$ of the symmetric stiffness matrix $\mathbf{K}$ is interpreted as a reactive moment appearing at the clamped joint $i$ of the structural member $ij$ as a result of the unit angle rotation at the clamped joint $j$.

Because of the fact that the values of reactive moments of a simple clamped beam under rotation of one of its ends are as two to one, the diagonal and non-diagonal entries of matrix $\mathbf{K}$ are simply related. Particularly, the diagonal entry is twice the sum of non-diagonal entries of the row (column). It can be written:

$$K_{ii} = \sum_{\substack{j=1 \\ i\neq j}}^{m} \overline{M}_{ij} = \sum_{\substack{j=1 \\ i\neq j}}^{m} 4\left(\frac{EI}{L}\right)_{ij}, \tag{2}$$

$$K_{ij} = \frac{1}{2}\overline{M}_{ij} = \frac{1}{2}\overline{M}_{ji} = 2\left(\frac{EI}{L}\right)_{ij}, \tag{3}$$

where $\overline{M}_{ij}$ is a reactive moment at the end of the element $ij$ corresponding to the $i$th nodal joint under unit rotation of this joint; $E$, $I$ and $L$ are the elasticity modulus, moment of inertia and length correspondingly of the $ij$th structural element. The reactive moment at the end of the element $ij$ corresponding to the $i$th nodal joint under unit rotation of the $j$th joint equals the reactive moment at the end of the element $ij$ corresponding to the $j$th nodal joint under unit rotation of the $i$th joint as is shown in the second equality of (3).

Eq. (2) is correct when the $i$th nodal joint is internal, that is there are not elements connecting this joint to support points. Otherwise, the diagonal entry $K_{ii}$ should be completed with the corresponding terms. In the case where the element at the $i$th node is hinged at the supporting point, then:

$$K_{ii} = \cdots + \overline{M}_{\text{in}} = \cdots + 3\left(\frac{EI}{L}\right)_{in}, \tag{4}$$

where '*in*' is the index of the "supporting" element. In the case where the '*in*' element at the $i$th node is clamped at the supporting point, then:

$$K_{ii} = \cdots + \overline{M}_{\text{in}} = \cdots + 4\left(\frac{EI}{L}\right)_{in}. \tag{5}$$

## 3. Jacobi solution scheme

Let Jacobi iterative procedure (Golub and Van Loan, 1996) is applied to the solution of Eq. (1). In this case it is obtained:

$$\overset{s+1}{\mathbf{u}} = -\mathbf{B}^{-1}(\mathbf{C}\overset{s}{\mathbf{u}} - \mathbf{p}), \tag{6}$$

where the upper index means the iteration number; matrix $\mathbf{B}$ comprises diagonal entries of matrix $\mathbf{K}$ ($\mathbf{B} = \mathrm{Diag}\,\mathbf{K}$) and matrix $\mathbf{C}$ properly obtained: $\mathbf{C} = \mathbf{K} - \mathbf{B}$. Componentwisely, Eq. (6) takes the form:

$$\overset{s+1}{u_i} = -K_{ii}^{-1}\sum_{\substack{j=1\\ j\neq i}}^{m}(K_{ij}\overset{s}{u_j} + p_i). \tag{7}$$

The initial guess is $\overset{0}{\mathbf{u}} = \mathbf{0}$. This way of starting the iterative procedure is of no influence on the convergence of the procedure because of the reasons discussed below. However, it is necessary to show relationships between Jacobi iterations and the method of Hardy Cross.

## 4. The Hardy Cross method

In order to show equivalence of Jacobi iterative scheme and the Cross method it is necessary to reformulate the iterative procedure described above in the incremental form. In this case Eq. (7) takes the form:

$$\overset{s+1}{\Delta u_i} = -K_{ii}^{-1}\sum_{\substack{j=1\\ j\neq i}}^{m}K_{ij}\overset{s}{\Delta u_j}, \qquad \overset{0}{\Delta u_i} = -K_{ii}^{-1}p_i, \qquad u_i = \sum_{s}\overset{s}{\Delta u_i}. \tag{8}$$

By pre-multiplying Eq. (8) by $\overline{M}_{ij}$ and by considering Eqs. (2) and (3) it takes the form:

$$\overset{s+1}{\Delta M_{ij}} = -\left[\frac{\overline{M}_{ij}}{\sum\limits_{\substack{j=1\\ j\neq i}}^{m}\overline{M}_{ij}}\right]\sum_{\substack{j=1\\ j\neq i}}^{m}\overset{s}{\Delta M_{ij}}/2, \tag{9}$$

where $\overset{s+1}{\Delta M_{ij}} = \overline{M}_{ij}\overset{s+1}{\Delta u_i}$ is the increment of the $ij$th element end moment at the $i$th joint; $\overset{s}{\Delta M_{ij}}/2 = K_{ij}\overset{s}{\Delta u_j}$ is the "carryover" moment increment transmitted to the $i$th joint from the $j$th joint as a result of the rotation of the latter on angle $\overset{s}{\Delta u_j}$. The coefficient in the brackets is the "distribution coefficient" of Hardy Cross.

Eq. (9) is the general formal notation of the Cross method. The number of Cross' equations for the $i$th node equals the number of the elements at this node; these equations, however, are not independent. Only one equation is independent. The rest are related as follows:

$$\frac{\overset{s+1}{\Delta M_{ij}}}{\overset{s+1}{\Delta M_{iy}}} = \frac{\overline{M}_{ij}}{\overline{M}_{iy}}. \tag{10}$$

Summarizing, it is possible to claim that the Cross method is an incremental form of Jacobi iterative scheme applied to the equilibrium equations in terms of displacements.

## 5. Convergence

Convergence of the Cross method or, equivalently, Jacobi iterative scheme is provided due to diagonal dominance of matrix **K**. The diagonal dominance is a direct consequence of the matrix construction. Indeed, as was pointed out before, the diagonal entry is twice the sum of non-diagonal entries of the row (column) for internal joints and even larger for joints connected to supporting points.

Under diagonal dominance Jacobi iterations converge for any loading conditions (see Golub and Van Loan, 1996). It is interesting to note that for such computationally attractive matrices as **K** there are other iterative schemes with faster convergence than Jacobi iterations; Gauss–Seidel iterative scheme for example.

## 6. Conclusions

- It was shown that the Hardy Cross moment-distribution method is Jacobi iterative scheme applied to the standard displacement formulation of structural analysis.
- It was shown that the stiffness matrix enjoys strict diagonal dominance and because of it the iterations always converge for any loading conditions.

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