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# Comments and authors' reply on "Linear stress-strain relations in nonlinear elasticity" by A. Chiskis and R. Parnes, (Acta Mech. 146, 109–113, 2001)

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**Summary.** Recently this journal published a work stating that the idea of geometrical nonlinearity within Hooke's law is "no more than a widely accepted illusion since the linear stress-strain laws hold only for very nontrivial measures representing the corresponding strain tensor which depend on material parameters". Since the linear stress-strain relations with nonlinear strains are, indeed, widely used in research and design the arguments of the authors of this work should be considered. Below it is shown where a flaw in these arguments is and why Hooke's law with nonlinear strains is correct.

## 1 Results of Chiskis and Parnes [1]

The following problem is considered in Chiskis and Parnes [1]. Let the left stretch tensor represent the basic deformation measure

$$\mathbf{V} = \lambda_1 \mathbf{v}_1 \otimes \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \otimes \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 \otimes \mathbf{v}_3, \tag{1}$$

where $\lambda_i$ designates principal stretches and $\mathbf{v}_n$ are the proper orthonormal vectors in the current configuration.

Assume that the strain measure $\mathbf{A}$ is a nonlinear function of $\mathbf{V}$:

$$\mathbf{A} = \hat{\mathbf{A}}(\mathbf{V}) = f(\lambda_1)\mathbf{v}_1 \otimes \mathbf{v}_1 + f(\lambda_2)\mathbf{v}_2 \otimes \mathbf{v}_2 + f(\lambda_3)\mathbf{v}_3 \otimes \mathbf{v}_3. \tag{2}$$

This strain measure together with the Cauchy stress tensor $\mathbf{T}$ define the generalized Hooke's law

$$\mathbf{T} = \Lambda(\mathbf{A} : \mathbf{1})\mathbf{1} + 2\mu\mathbf{A}, \tag{3}$$

where $\mathbf{1}$ is the second-order identity tensor, while $\Lambda$ and $\mu$ are Lame's coefficients.

Let, finally, the strain energy exist,

$$W = \hat{W}(\lambda_1, \lambda_2, \lambda_3). \tag{4}$$

The question is: *What restrictions should be imposed on $f(\lambda_i)$ to provide the existence of the strain energy*? The answer appears as follows.

Equation (3) can be rewritten in principal values,

$$T_i = \Lambda(A_1 + A_2 + A_3) + 2\mu A_i, \tag{5}$$

$$A_i = f(\lambda_i). \tag{6}$$

The Einstein summation convention is suppressed henceforth.

The relation between the principal values of the Cauchy stress and the strain energy can be written as

$$T_i = \frac{\lambda_i}{J}\frac{\partial W}{\partial \lambda_i}, \tag{7}$$

$$J = \lambda_1 \lambda_2 \lambda_3. \tag{8}$$

From Eq. (7) we obtain

$$\frac{\partial W}{\partial \lambda_i} = T_i \frac{J}{\lambda_i}. \tag{9}$$

Now the compatibility condition is formulated in the form

$$\frac{\partial}{\partial \lambda_j}\left(T_i \frac{J}{\lambda_i}\right) = \frac{\partial}{\partial \lambda_i}\left(T_j \frac{J}{\lambda_j}\right). \tag{10}$$

We remind again that there is no summation over the repeated indices.

Substituting from Eqs. (5) and (6) in Eq. (10) and solving it we obtain

$$A_i = f(\lambda_i) = C_1 \lambda_i^{2\mu/\Lambda} + C_2. \tag{11}$$

The unknown constants can be determined from the condition that measure $A_i$ equals the classical linear measure for $\lambda_i \approx 1$:

$$\varepsilon_i = \lambda_i - 1, \tag{12}$$

Expanding Eq. (11) in a power series about $\lambda_i = 1$ we have

$$A_i = f(\lambda_i) = C_1 + C_2 + C_1 \frac{2\mu}{\Lambda}(\lambda_i - 1) + O((\lambda_i - 1)^2). \tag{13}$$

Comparing Eqs. (12) and (13) we conclude

$$C_2 = -C_1,\, C_1 = \frac{\Lambda}{2\mu}, \tag{14}$$

and

$$A_i = f(\lambda_i) = \frac{\Lambda}{2\mu}(\lambda_i^{2\mu/\Lambda} - 1). \tag{15}$$

The latter can be written in the general form

$$\mathbf{A} = \frac{\Lambda}{2\mu}(\mathbf{V}^{2\mu/\Lambda} - \mathbf{1}). \tag{16}$$

This result means that no purely geometrical strain measure exists in general. This is the reason for the authors' general reservations on the use of the nonlinear strains cited above.

## 2 Where is the flaw?

Consider Hooke's law with the second Piola-Kirchhoff stress, instead of the Cauchy stress,

$$S_i = \Lambda(A_1 + A_2 + A_3) + 2\mu A_i. \tag{17}$$

The relation between the principal values of the second Piola-Kirchhoff stress and the strain energy takes the form

$$S_i = \frac{1}{\lambda_i}\frac{\partial W}{\partial \lambda_i} \text{ (No sum!).} \tag{18}$$

The compatibility condition is written as

$$\frac{\partial}{\partial \lambda_j}(S_i\lambda_i) = \frac{\partial}{\partial \lambda_i}(S_j\lambda_j). \tag{19}$$

Substituting from Eqs. (17) and (6) in Eq. (19) we have

$$\frac{\partial f(\lambda_i)}{\lambda_i \partial \lambda_i} = \frac{\partial f(\lambda_j)}{\lambda_j \partial \lambda_j}. \tag{20}$$

Integrating this equation we get

$$A_i = f(\lambda_i) = \frac{C_1}{2}\lambda_i^2 + C_2. \tag{21}$$

The unknown constants can again be determined from the condition that the measure $A_i$ equals the classical linear measure for $\lambda_i \approx 1$ . Expanding Eq. (21) in a power series about $\lambda_i = 1$, we have

$$A_i = f(\lambda_i) = \frac{C_1}{2} + C_2 + C_1(\lambda_i - 1) + O((\lambda_i - 1)^2). \tag{22}$$

Comparing Eqs. (12) and (22) we conclude

$$C_1 = 1,\ C_2 = -\frac{1}{2}, \tag{23}$$

and

$$A_i = f(\lambda_i) = \frac{1}{2}(\lambda_i^2 - 1) = E_i. \tag{24}$$

The latter can be written in the general form

$$\mathbf{A} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \mathbf{E}, \tag{25}$$

where **F** is the deformation gradient and **E** is the Green strain.

Thus we have a purely geometrical nonlinear strain that is entirely compatible with the existence of the strain energy. The latter is readily written as

$$W = \frac{\Lambda}{8}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^2 + \frac{\mu}{4}[(\lambda_1^2 - 1)^2 + (\lambda_2^2 - 1)^2 + (\lambda_3^2 - 1)^2]. \tag{26}$$

Substituting Eq. (26) in Eq. (18) we have

$$S_i = \frac{\Lambda}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \mu(\lambda_i^2 - 1). \tag{27}$$

This is nothing but Hooke's law. The "geometrical nonlinearity" is correct in this case.

In summary, the Cauchy stress does not allow for a nonlinear strain, which is compatible with the existence of the strain energy, while the second Piola-Kirchhoff stress does allow for the energy compatible nonlinear strain. Why? We turn back to Eq. (18) to answer this question. This equation can be rewritten in the form

$$S_i = \sum_j \frac{\partial W}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial E_i} = \frac{\partial W}{\partial E_i}. \tag{28}$$

This is possible because (see Eq. (24))

$$\frac{\partial \lambda_j}{\partial E_i} = \begin{cases} \lambda_i^{-1}, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \tag{29}$$

Thus the second Piola-Kirchhoff stress is derived from the strain energy by the direct differentiation with respect to the Green strain. This is possible iff the stress is work-conjugate to the strain. The latter means that the double contraction of the stress tensor with the material time derivative of the strain tensor equals the work rate (the stress power rate).

The Cauchy stress tensor does not enjoy a work-conjugate strain tensor. This fact, which is known in the literature (Ogden [2]), is reflected in the results of Chiskis and Parnes [1].

It is worth mentioning, finally, that for practical purposes one should not begin the construction of Hooke's law with the choice of the stress measure. On the contrary, a nonlinear strain measure **A** should be chosen first. In this case the strain energy takes the form

$$W = \frac{1}{2}\Lambda(\mathbf{A} : \mathbf{1})^2 + \mu\mathbf{A} : \mathbf{A}, \tag{30}$$

and the stress is introduced *by definition* as

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \mathbf{A}}. \tag{31}$$

In this way the compatibility of the nonlinear strain with the strain energy is always guaranteed.

## References

[1] Chiskis, A., Parnes, R.: Linear stress-strain relations in nonlinear elasticity. Acta Mech. **146,** 109–103 (2001).
[2] Ogden, R. W.: Nonlinear elastic deformations. Chichester: Ellis Horwood 1984.

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# Authors' reply

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In the article appearing in Acta Mechanica on the use of Hooke's law in nonlinear elasticity, Chiskis and Parnes [1] showed that it is not possible to establish a universal strain measure independent of the Lamé constants in nonlinear elasticity and that only strain measures dependent on specific values of Poisson's ratio are valid.

In the above comment, Volokh [2] disagrees with these results and claims that the results of the original paper contain a flaw. It should be emphasized that the original article [1] is concerned exclusively with the Cauchy stress tensor. However, the discovery in [2] of the purported flaw is then based on a development using the second Piola-Kirchhoff stress tensor. Now, clearly, the two stress tensors are of different nature: the Cauchy stress tensor is based on the current configuration and is the fundamental physical tensor; to quote Lurie [3, Chap. 2, Sect. 6]: "The description of the state of stress using the Cauchy tensor, determined in the actual configuration is natural and physically obvious'', while further it is stated in [3] that the Piola tensor and its combinations are ''... convenient auxiliary quantities which, however, do not directly determine the state of stress. To this end it is necessary to return to the Cauchy tensor **T** which is sometimes called the true stress tensor''. Thus it is clear that the Piola-Kirchhoff stress tensor cannot be directly connected to the traction and normal vector, but instead the use of a geometrically nonlinear transformation to obtain the true Cauchy stress tensor is required.

To summarize: while a linear stress-strain relation is indeed obtained in [2] for the Piola-Kirchhoff stress tensor, the author unfortunately disregards the need of such a geometrically nonlinear transformation to obtain the true Cauchy stress tensor. The assumed existence of the purported flaw is thus entirely specious.

One may therefore finally assert, as originally shown in [1], that the linear Hooke's law using the Cauchy stress tensor is not universally possible in nonlinear elasticity but is valid only for specific strain measures which are dependent on values of Poisson's ratio.

## References

[1] Chiskis, A., Parnes, R.: Linear stress-strain relations in nonlinear elasticity. Acta Mech. **146**, 109–113 (2001).
[2] Volokh, K. Y.: Comments on ''Linear stress-strain relations in nonlinear elasticity'', by A. Chiskis and R. Parnes [Acta Mech. 146, 109–113, 2001]. Acta Mech. **171**, 243–244 (2004).
[3] Lurie, A. I.: Nonlinear theory of elasticity. Amsterdam: North-Holland 1990.

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