

Non-linear thermoelasticity with energy limiters



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ABSTRACT

In this article we couple the previously developed theory of elasticity with energy limiters for modeling material failure with a description of thermal processes. We present a generalized formulation of the thermoelasticity with energy limiters. We illustrate the new theory via the analytically tractable example of uniaxial tension. The theory predicts the existence of the thermoelastic inversion point observed in experiments. Besides, it predicts the experimentally observed stiffening of rubberlike materials under heating. We find above all that the tensile strength increases as a result of the stiffening while the critical rupture stretch is almost unaffected by heating. It is hoped that the developed coupled theory can be useful for modeling failure in rubberlike solids.

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1. Introduction

Continuum thermodynamics has a long history yet its formulation is still open to debate [11,21,22,13,29,12,15,7]. We will not participate in the debate, however. Instead, we will consider the simplest formulation that couples thermal and mechanical processes in the course of large deformations of rubberlike solids. The latter theory of non-linear thermoelasticity was pioneered by Chadwick [3] and further considered in Refs. [4,16,14,9,1,17], for example.

The existing theories of thermo-mechanics of rubberlike solids consider the intact material behavior, in which the deformation description does not incorporate mechanical failure. However, real materials do fail and their constitutive equations should include a description of failure. In the context of purely mechanical deformation without thermal coupling a very simple account of material failure in the constitutive laws was proposed in a series of recent publications: [23,24,25,27,28]. The basic idea was to introduce an energy limiter in the expression for strain energy. Such limiter enforces saturation – *the failure energy* – in the strain energy function, which indicates the maximum amount of energy that can be stored and dissipated by an infinitesimal material volume during rupture. The limiter induces stress bounds in the constitutive equations *automatically*.

The mentioned approach of elasticity with energy limiters does not include the thermo-mechanical coupling. The purpose of the present work is to fill this gap and propose a variant of the non-linear thermoelastic formulation with energy limiters. It is worth emphasizing that such formulation is by no means unique and the

present one can be seen as a starting point. This new formulation is given in Section 3 after a brief review of non-linear thermoelasticity in Section 2. Then the account of thermoelastic incompressibility is given in Section 4. The latter issue was considered more or less implicitly in the previous literature and it is reasonable, perhaps, to introduce it explicitly and separately. In Sections 5 and 6 respectively we further specialize the constitutive model and apply it to the problem of homogeneous uniaxial tension that is tractable analytically. Discussion in Section 6 summarizes the work.

We mention, finally, that the developed theory predicts the existence of the thermoelastic inversion point observed in experiments. Besides, it predicts the experimentally observed stiffening of rubberlike materials under heating. Remarkably, we find that the tensile strength increases as a result of the stiffening while the critical stretch of rupture is almost unaffected by heating. It is hoped that the developed coupled theory can be useful for modeling failure in rubberlike solids.

2. Basic finite thermoelasticity

Material point that occupies position \mathbf{x} in the reference configuration moves to position $\mathbf{y}(\mathbf{x})$ in the current configuration of a deformable body Ω . Deformation in the vicinity of the material point is described by the deformation gradient tensor

$$\mathbf{F} = \text{Grad } \mathbf{y}. \quad (2.1)$$

The linear and angular momentum balance take the following forms accordingly:

$$\text{Div } \mathbf{P} + \mathbf{b} = \rho \mathbf{a}, \quad (2.2)$$

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$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T, \quad (2.3)$$

where the divergence operator is calculated with respect to referential coordinates \mathbf{x} ; \mathbf{P} is the 1st Piola–Kirchhoff stress tensor; \mathbf{b} is the body force pure unit reference volume; and ρ and \mathbf{a} are the referential mass density and acceleration vector correspondingly.

Balance of linear momentum on the body surface $\partial\Omega$ reads

$$\mathbf{P}\mathbf{n} = \bar{\mathbf{t}}, \quad (2.4)$$

where $\bar{\mathbf{t}}$ is a prescribed traction per unit area of the reference surface with the unit outward normal \mathbf{n} .

Alternatively to (2.4) a surface boundary condition can be imposed on placements

$$\mathbf{y} = \bar{\mathbf{y}}, \quad (2.5)$$

where the barred quantity is prescribed on the surface $\partial\Omega$.

Initial conditions are

$$\mathbf{y}(t=0) = \mathbf{y}_0, \quad \mathbf{v}(t=0) = \mathbf{v}_0, \quad (2.6)$$

where \mathbf{v} is the velocity vector and \mathbf{y}_0 and \mathbf{v}_0 are prescribed in Ω .

Equation of the energy balance takes form

$$\dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \text{Div } \mathbf{Q} + R, \quad (2.7)$$

where e is the internal energy per unit reference volume; \mathbf{Q} is the Piola–Kirchhoff heat flux; and R is a heat source.

Instead of the internal energy we use the Helmholtz free energy per unit reference volume

$$\psi(\mathbf{F}, \theta) = e - \theta\eta, \quad (2.8)$$

where η is the entropy per unit reference volume and θ is the absolute temperature.

Differentiating (2.8) with respect to time we obtain

$$\frac{\partial\psi}{\partial\mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial\psi}{\partial\theta}\dot{\theta} = \dot{e} - \dot{\theta}\eta - \theta\dot{\eta}. \quad (2.9)$$

Substitution of the rate of the internal energy from (2.9) into (2.7) yields the energy balance in the form

$$\theta\dot{\eta} = (\mathbf{P} - \frac{\partial\psi}{\partial\mathbf{F}}) : \dot{\mathbf{F}} - (\eta + \frac{\partial\psi}{\partial\theta})\dot{\theta} - \text{Div } \mathbf{Q} + R. \quad (2.10)$$

Let us restrict further considerations by the following general constitutive laws:

$$\mathbf{P} = \frac{\partial\psi}{\partial\mathbf{F}}, \quad (2.11)$$

$$\eta = -\frac{\partial\psi}{\partial\theta}. \quad (2.12)$$

We note that in this case the entropy rate is

$$\dot{\eta} = -\frac{\partial^2\psi}{\partial\theta\partial\mathbf{F}} : \dot{\mathbf{F}} - \frac{\partial^2\psi}{\partial\theta^2}\dot{\theta}. \quad (2.13)$$

Substitution of (2.11)–(2.13) in (2.10) finally yields

$$c\dot{\theta} = -\text{Div } \mathbf{Q} + \theta \frac{\partial^2\psi}{\partial\theta\partial\mathbf{F}} : \dot{\mathbf{F}} + R, \quad (2.14)$$

where

$$c = -\theta \frac{\partial^2\psi}{\partial\theta^2} > 0, \quad (2.15)$$

is the specific heat capacity.

Balance of energy on the body surface $\partial\Omega$ reads

$$-\mathbf{Q}\mathbf{n} = \bar{\mathbf{q}}, \quad (2.16)$$

where $\bar{\mathbf{q}}$ is a prescribed heat flux through the unit area of the reference surface.

Alternatively to (2.16) a surface boundary condition can be imposed on temperature

$$\theta = \bar{\theta}. \quad (2.17)$$

Initial condition on the temperature might be necessary for the transition processes

$$\theta(t=0) = \theta_0, \quad (2.18)$$

where θ_0 is prescribed in Ω .

To complete the boundary value problem it remains to formulate the heat conduction equation in the Duhamel form, for example,

$$\mathbf{Q} = -J \mathbf{F}^{-1} \boldsymbol{\kappa} \mathbf{F}^{-T} \text{Grad } \theta, \quad (2.19)$$

where $J = \det\mathbf{F}$ and $\boldsymbol{\kappa}$ is the spatial thermal conductivity tensor.

We note that (2.19) is a generalization of the Fourier's constitutive law of heat conduction – see, for example, Holzapfel [9].

We note also that the second law of thermodynamics (in the Clausius–Planck form) requires the thermo-elastic processes to obey the following constraints:

$$D = \mathbf{P} : \dot{\mathbf{F}} - \dot{\psi} - \eta\dot{\theta} \geq 0, \quad (2.20)$$

$$\mathbf{Q} \cdot \text{Grad } \theta \leq 0. \quad (2.21)$$

The first constraint – the dissipation inequality – is obeyed by constitutive laws (2.11)–(2.12). The second constraint is obeyed by constitutive law (2.19) in which the spatial thermal conductivity tensor is positively definite.

3. Finite thermoelasticity with energy limiters

In this section, we incorporate a failure description in the non-linear thermoelasticity by using energy limiters. We start with one energy limiter. In this case the Helmholtz free energy per unit reference volume takes form

$$\psi(\mathbf{F}, \theta, \alpha) = \psi_f - H(\alpha) \psi_{te}(\mathbf{F}, \theta), \quad (3.1)$$

$$\psi_f = \psi_{te}(\mathbf{1}, \theta_0), \quad (3.2)$$

$$\psi_{te}(\mathbf{F}, \theta) \rightarrow 0, \quad \text{when } \|\mathbf{F}, \theta\| \rightarrow \infty, \quad (3.3)$$

where ψ_f and $\psi_{te}(\mathbf{F}, \theta)$ designate a constant bulk failure energy and a thermoelastic free energy respectively; $H(z)$ is a unit step function, i.e. $H(z) = 0$ if $z < 0$ and $H(z) = 1$ otherwise; $\mathbf{1}$ is a second-order identity tensor; and $\|\mathbf{F}, \theta\| = \mathbf{F} : \mathbf{F} + \theta^2$, for example.

The switch parameter $\alpha \in (-\infty, 0]$ is defined by the evolution equation

$$\dot{\alpha} = -H(\varepsilon - \psi_{te}/\psi_f), \quad \alpha(t=0) = 0 \quad (3.4)$$

where $0 < \varepsilon < 1$ is a dimensionless precision constant.

The physical interpretation of (3.1)–(3.4) is straight: material response is thermoelastic as long as the free energy is below its limit, ψ_f . When the limit is reached, then the free energy remains constant for the rest of the deformation process, thereby making material healing impossible. Parameter α is not an internal damage variable (like in damage mechanics) and it functions as a switch: if $\alpha = 0$ the process is thermoelastic, and if $\alpha < 0$, the material is irreversibly damaged and the thermoelastic energy is dissipated.

In presence of the switch parameter α , the free energy increment reads

$$\dot{\psi}(\mathbf{F}, \theta, \alpha) = \frac{\partial\psi}{\partial\mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\alpha}\dot{\alpha}. \quad (3.5)$$

The last term in (3.5) can be calculated based on (3.1)

$$\frac{\partial\psi}{\partial\alpha}\dot{\alpha} = -\delta(\alpha)\dot{\alpha}\psi_{te}, \quad (3.6)$$

where $\delta(\alpha)$ is the Dirac delta.

We notice that (3.6) in fact vanishes for all values of α . The case of $\alpha \neq 0$ follows immediately from the definition of Dirac's delta. In

the case of $\alpha = 0$, we note that the relation $\alpha = \int \dot{\alpha} dt = 0$ along with the definition of $\dot{\alpha}$ as a step function imply $\dot{\alpha} \equiv 0$ since a non-positive integrand must be identically zero for the integral to vanish.

Thus, we have

$$\frac{\partial \psi_f}{\partial \alpha} \dot{\alpha} = -\delta(\alpha) \dot{\alpha} \psi_{te} = 0, \tag{3.7}$$

and all formulae presented in the previous section remain valid and do not require corrections concerning the switch parameter.

In order to enforce the energy limiter in the free Helmholtz energy function, we use the following expression for thermoelastic free energy:

$$\psi_{te}(\mathbf{F}, \theta) = \Phi m^{-1} \Gamma(m^{-1}, W^m(\mathbf{F}, \theta) \Phi^{-m}), \tag{3.8}$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$ is the upper incomplete gamma function; $W(\mathbf{F}, \theta)$ is the thermoelastic free energy of intact (without failure) material; Φ is the energy limiter, which is calibrated in macroscopic experiments; and m is a dimensionless material parameter, which controls sharpness of the transition to material failure on the stress–strain curve. Increasing or decreasing m it is possible to simulate more or less steep ruptures of the internal bonds accordingly.

The failure energy can be calculated via (3.2) as follows:

$$\psi_f = \Phi m^{-1} \Gamma(m^{-1}, W^m(\mathbf{1}, \theta_0) \Phi^{-m}). \tag{3.9}$$

Substitution of (3.8)–(3.9) in (3.1) and (2.9)–(2.10) yields

$$\mathbf{P} = -H(\alpha) \frac{\partial \psi_{te}}{\partial \mathbf{F}} = \exp(-W^m \Phi^{-m}) H(\alpha) \frac{\partial W}{\partial \mathbf{F}}, \tag{3.10}$$

$$\eta = H(\alpha) \frac{\partial \psi_{te}}{\partial \theta} = -\exp(-W^m \Phi^{-m}) H(\alpha) \frac{\partial W}{\partial \theta}. \tag{3.11}$$

For the sake of illustration and for the mechanical process at the room temperature we mention the following specialization of (3.10) for a filled Natural Rubber (NR) vulcanizate [30,22]

$$W = \sum_{k=1}^3 c_k (I_1 - 3)^k, \quad I_1 = \mathbf{F} : \mathbf{F}, \quad J = 1, \quad m = 10, \tag{3.12}$$

where $c_1 = 0.298$ MPa, $c_2 = 0.014$ MPa, $c_3 = 0.00016$ MPa, $\Phi = 82.0$ MPa.

The Cauchy stress ($\boldsymbol{\sigma} = J^{-1} \mathbf{P} \mathbf{F}^T$)–stretch curve for the NR model described by (3.9) and (3.12) is shown in Fig. 1, where also the results are shown for the intact model ($\Phi \rightarrow \infty$). Material failure occurs at the limit point at critical stretch $\lambda_{cr} = 7.12$ in accordance with experimental data by Hamdi et al. [8].

Consequences of the elasticity with energy limiters formulation are reviewed in [27]. Here, we only mention two theoretical

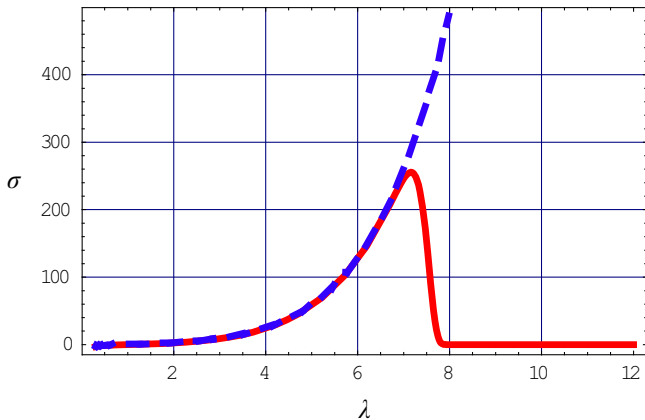


Fig. 1. Cauchy stress [MPa] versus stretch in uniaxial tension of NR; dashed line designates the intact model; solid line designates the model with energy limiter.

predictions that can be compared to the experimental data. Fig. 2a presents the critical failure stretches in a thin sheet of NR undergoing biaxial tension with different biaxiality ratios. Predictions based on the softening hyperelasticity model are compared to the reported test results. The theoretical model was calibrated in uniaxial tension discussed above and somewhat lower critical stretches in equal biaxial tension are expected in view of the high imperfection sensitivity of the experiments. Fig. 2b presents a cross-section of a natural rubber specimen in the “poker-chip” test [6]. The cut was done at the hydrostatic tension of ~ 2.7 MPa. The grown spherical cavities are visible. Softening hyperelasticity model [26] predicts the onset of instability and growth of the microscopic pre-existing cavities at the hydrostatic tension of ~ 2.4 MPa. Both comparisons with the experimental data encourage the use of the methods of energy limiters.

In the case where more than one energy limiter is necessary [28] the free energy function can be split into a sum as follows:

$$\psi = \sum_{n=1}^N \psi_n, \tag{3.13}$$

$$\psi_n(\mathbf{F}, \theta, \alpha_n) = \psi_{f,n} - H(\alpha_n) \psi_{te,n}(\mathbf{F}, \theta), \tag{3.14}$$

$$\dot{\alpha}_n = -H(\epsilon_n - \psi_{te,n} / \psi_{f,n}), \quad \alpha_n(t=0) = 0, \tag{3.15}$$

where the n th component represents a constituent, characteristic direction etc.

We note that according to (3.13)–(3.15) failure of the n th constituent does not lead to overall failure.

It is possible to modify the formulation in such a way that failure of any constituent will lead to the overall failure. This can be achieved for instance as follows:

$$\psi = \sum_{n=1}^N \beta_n \psi_n, \quad \beta_n = \prod_{\substack{j=1 \\ j \neq n}}^N H(\alpha_j), \tag{3.16}$$

Therefore, we can derive the corresponding constitutive equations

$$\mathbf{P} = - \prod_{j=1}^N H(\alpha_j) \sum_{n=1}^N \frac{\partial \psi_{te,n}}{\partial \mathbf{F}}, \tag{3.17}$$

$$\eta = \prod_{j=1}^N H(\alpha_j) \sum_{n=1}^N \frac{\partial \psi_{te,n}}{\partial \theta}. \tag{3.18}$$

Remark 1

We note that the account of dissipation via step function is necessary when the material unloading is sound as in the case of crack propagation, for example. Otherwise, the step function can be dropped from equations.

4. Thermoelastic incompressibility

In this section, we discuss the thermoelastic incompressibility constraint. The assumption of thermoelastic incompressibility can be written in the following form:

$$J = \det \mathbf{F} = g(\theta), \tag{4.1}$$

$$g(\theta_0) = 1, \tag{4.2}$$

which means that material is incompressible in the reference state and all volumetric changes are produced by the temperature alterations.

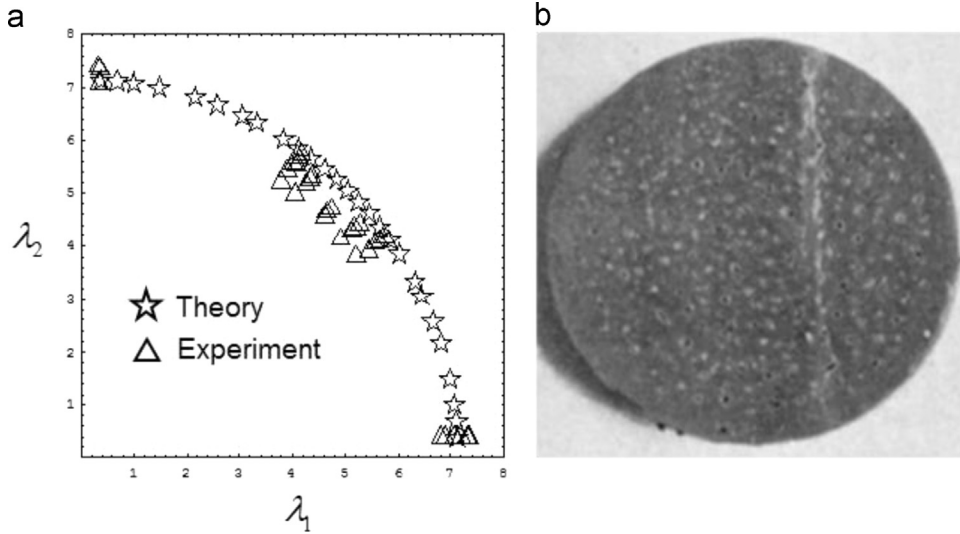


Fig. 2. (a) Critical failure stretches in biaxial tension for natural rubber and (b) grown cavities in the poker-chip test.

The time increment of this constraint takes form

$$\frac{\partial g}{\partial \theta} \dot{\theta} - J \mathbf{F}^{-T} : \dot{\mathbf{F}} = 0. \quad (4.3)$$

Here multipliers $\partial g / \partial \theta$ and $-J \mathbf{F}^{-T}$ represent workless entropy and stress accordingly, which can be scaled by arbitrary factor Π . With account of (4.3) we modify constitutive Eqs. (2.11) and (2.12) as follows:

$$\mathbf{P} = \frac{\partial \psi}{\partial \mathbf{F}} - J \mathbf{F}^{-T} \Pi, \quad (4.4)$$

$$\eta = -\frac{\partial \psi}{\partial \theta} + \frac{\partial g}{\partial \theta} \Pi. \quad (4.5)$$

For example, a general non-linear constraint obeying (4.2) can be defined as

$$g = \exp[3\gamma_0(\theta - \theta_0)], \quad (4.6)$$

$$\frac{\partial g}{\partial \theta} = 3\gamma_0 \exp[3\gamma_0(\theta - \theta_0)], \quad (4.7)$$

where γ_0 is a constant thermal expansion coefficient.

We note finally that the explicit notion of thermoelastic incompressibility is useful for analytical solutions of boundary value problems. When a numerical solution is required the thermoelastic incompressibility should be handled by different approaches. For example, it can be penalized by large bulk modulus and the like.

4.1. Remark 2

Chadwick and Scott [5] found that one of four solutions for the problem of the propagation of a plane acoustic wave in a constrained thermoelastic material was physically unreasonable. Based on this observation the authors concluded that the whole theory of constrained thermoelasticity was questionable. Such a conclusion is open to debate. Among formal mathematical solutions there can always be physically unreasonable and they should be filtered out. The classical example in solid mechanics is a solution for axisymmetric bending of a thin circular plate. When solved in polar coordinates the bi-harmonic equation describing the plate bending possesses a singular solution (logarithmic singularity). This solution is filtered out and not the theory underlying it.

It is interesting to note that Scott [18] reformulated the constrained thermoelastic theory in order to exclude the physically unreasonable solution in advance. It was remarkable that the revised formulation, based on the deformation-entropy constraint, allowed for only three (instead of four) solutions of the coupled problem. Unfortunately, the revised formulation required abandoning the natural use of the temperature in the thermoelastic constraint. The latter price (in the present author's opinion) is probably too high to be paid for filtering out the physically unreasonable solution.

5. Model specialization

In this section, we specialize a coupled constitutive model with energy limiter for natural rubber vulcanizate. Thermoelastic free energy is defined as

$$\psi_{te} = 0.1 \Phi \Gamma(0.1, W^{10} \Phi^{-10}), \quad (5.1)$$

$$W = \frac{\theta}{\theta_0} \sum_{k=1}^3 c_k (I_1 - 3)^k + c_0 \left(\theta - \theta_0 - \theta \ln \frac{\theta}{\theta_0} \right), \quad (5.2)$$

where $c_1 = 0.298$ MPa, $c_2 = 0.014$ MPa, $c_3 = 0.00016$ MPa, $\Phi = 82.0$ MPa from the purely mechanical model described above and c_0 is a positive and constant specific heat capacity.

The derivatives of the Helmholtz free energy can be calculated now

$$\frac{\partial \psi}{\partial \mathbf{F}} = 2H(\alpha) \exp\left(-\frac{W^m}{\Phi^m}\right) \frac{\theta}{\theta_0} (c_1 + 2c_2(I_1 - 3) + 3c_3(I_1 - 3)^2) \mathbf{F}, \quad (5.3)$$

$$\frac{\partial \psi}{\partial \theta} = -H(\alpha) \exp\left(-\frac{W^m}{\Phi^m}\right) \left(\frac{1}{\theta_0} \sum_{k=1}^3 c_k (I_1 - 3)^k - c_0 \ln \frac{\theta}{\theta_0} \right), \quad (5.4)$$

$$\begin{aligned} \frac{\partial^2 \psi}{\partial \theta \partial \mathbf{F}} = & 2H(\alpha) \left\{ c_1 + 2c_2(I_1 - 3) + 3c_3(I_1 - 3)^2 \right\} \exp\left(-\frac{W^m}{\Phi^m}\right) \\ & \times \frac{1}{\theta_0} \left\{ 1 - \theta \frac{mW^{m-1}}{\Phi^m} \left(\frac{1}{\theta_0} \sum_{k=1}^3 c_k (I_1 - 3)^k - c_0 \ln \frac{\theta}{\theta_0} \right) \right\} \mathbf{F}, \end{aligned} \quad (5.5)$$

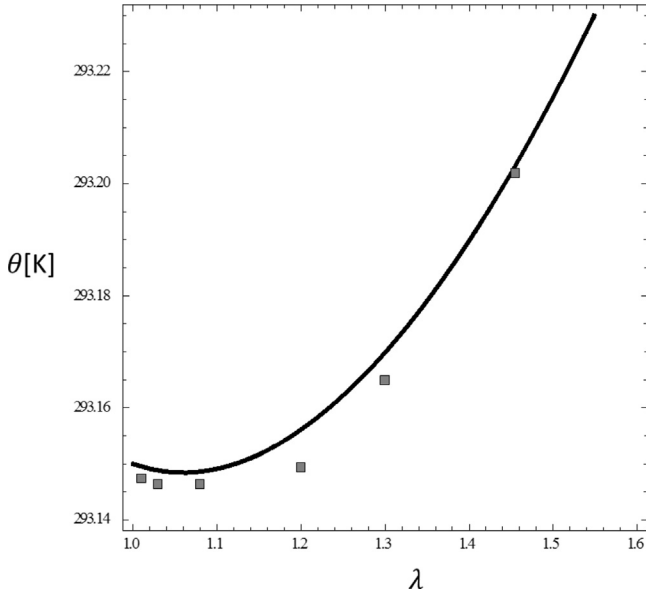


Fig. 3. Temperature versus stretch for constant entropy; thermoelastic inversion point at $\lambda_{inv} = 1.062$; filled squares show the Joule experimental data (taken from [9]).

$$\frac{\partial^2 \Psi}{\partial \theta^2} = H(\alpha) \exp\left(-\frac{W^m}{\Phi^m}\right) \frac{1}{\theta} \left\{ c_0 + \frac{mW^{m-1}}{\Phi^m} \left(\frac{1}{\theta_0} \sum_{k=1}^3 c_k (I_1 - 3)^k - c_0 \ln \frac{\theta}{\theta_0} \right)^2 \right\}. \quad (5.6)$$

Thus, constitutive Eqs. (4.4)–(4.5) accounting for thermoelastic incompressibility take forms

$$\mathbf{P} = 2H(\alpha) \exp\left(-\frac{W^m}{\Phi^m}\right) \frac{\theta}{\theta_0} (c_1 + 2c_2(I_1 - 3) + 3c_3(I_1 - 3)^2) \mathbf{F} - J \mathbf{F}^{-T} \Pi, \quad (5.7)$$

$$\eta = H(\alpha) \exp\left(-\frac{W^m}{\Phi^m}\right) \left(\frac{1}{\theta_0} \sum_{k=1}^3 c_k (I_1 - 3)^k - c_0 \ln \frac{\theta}{\theta_0} \right) + \frac{\partial g}{\partial \theta} \Pi. \quad (5.8)$$

We further assume that the spatial thermal conductivity tensor is spherical for isotropic material

$$\boldsymbol{\kappa} = \kappa_0(\theta) \mathbf{1}, \quad (5.9)$$

where $\kappa_0(\theta)$ is a scalar thermal conductivity.

Consequently, the constitutive equation of heat conduction (2.19) takes form

$$\mathbf{Q} = -J \kappa_0 (\mathbf{F}^T \mathbf{F})^{-1} \text{Grad } \theta. \quad (5.10)$$

6. Uniaxial tension

We consider uniaxial tension without heat sources ($R = 0$) and with spatially homogeneous temperature and deformations fields. The latter notion has strong implications. Indeed, in the absence of spatial gradient of the temperature field: $\text{Grad } \theta = \mathbf{0}$; there is no heat flux

$$\mathbf{Q} = \mathbf{0}. \quad (6.1)$$

In this case the energy balance Eq. (2.10) reduces to

$$\dot{\eta} = 0, \quad \eta = \text{constant}. \quad (6.2)$$

Assuming also that the body and inertia forces vanish and with account of the homogeneous deformation field we conclude that the first Piola–Kirchhoff stress is also homogeneous. Indeed, the

deformation can be written as follows:

$$\mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 + (J/\lambda)^{1/2} (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \quad (6.3)$$

where λ is the axial stretch and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are Cartesian base vectors.

Then, we obtain

$$I_1 = \lambda^2 + 2J/\lambda, \quad (6.4)$$

and the constitutive equation for stresses (5.7) takes form

$$\begin{aligned} P_1 &= 2 \exp\left(-\frac{W^m}{\Phi^m}\right) \frac{\theta}{\theta_0} (c_1 + 2c_2(\lambda^2 + 2J/\lambda - 3) \\ &\quad + 3c_3(\lambda^2 + 2J/\lambda - 3)^2) \lambda - \Pi \frac{J}{\lambda} \\ P_2 &= P_3 = 2 \exp\left(-\frac{W^m}{\Phi^m}\right) \frac{\theta}{\theta_0} (c_1 + 2c_2(\lambda^2 + 2J/\lambda - 3) \\ &\quad + 3c_3(\lambda^2 + 2J/\lambda - 3)^2) (J/\lambda)^{1/2} - J \Pi (J/\lambda)^{-1/2}, \end{aligned} \quad (6.5)$$

where P_i s are the axial (and principal) components of the 1st Piola–Kirchhoff stress tensor and the free energy of the intact material is

$$W = \frac{\theta}{\theta_0} \sum_{k=1}^3 c_k (\lambda^2 + 2J/\lambda - 3)^k + c_0 \left(\theta - \theta_0 - \theta \ln \frac{\theta}{\theta_0} \right). \quad (6.6)$$

We note that we do not consider unloading in this section and, thus, the step function is dropped from the equations for the sake of simplicity.

Since the lateral stresses vanish in uniaxial tension we can find the Lagrange parameter Π by equating (6.5)₂ to zero

$$\Pi = \frac{2}{\lambda} \exp\left(-\frac{W^m}{\Phi^m}\right) \frac{\theta}{\theta_0} [c_1 + 2c_2(\lambda^2 + 2J/\lambda - 3) + 3c_3(\lambda^2 + 2J/\lambda - 3)^2]. \quad (6.7)$$

Then, substituting the found Lagrange parameter in (6.5)₁ we obtain

$$\begin{aligned} P = P_1 &= 2 \exp\left(-\frac{W^m}{\Phi^m}\right) \frac{\theta}{\theta_0} (c_1 + 2c_2(\lambda^2 + 2J/\lambda - 3) \\ &\quad + 3c_3(\lambda^2 + 2J/\lambda - 3)^2) \left(\lambda - \frac{J}{\lambda^2} \right), \end{aligned} \quad (6.8)$$

or in terms of Cauchy stress

$$\boldsymbol{\sigma} = \frac{\lambda}{J} P. \quad (6.9)$$

In the limit case of $\theta = \theta_0$ we obtain $J = g(\theta_0) = 1$ and the stress–stretch curve (6.9) is presented in Fig. 1.

The constitutive equation for entropy (5.8) takes the following form, with account of (6.7),

$$\begin{aligned} \eta &= \exp\left(-\frac{W^m}{\Phi^m}\right) \left\{ \frac{1}{\theta_0} \sum_{k=1}^3 c_k (\lambda^2 + 2J/\lambda - 3)^k - c_0 \ln \frac{\theta}{\theta_0} \right. \\ &\quad \left. + 3\gamma_0 \exp[3\gamma_0(\theta - \theta_0)] \frac{2}{\lambda} \frac{\theta}{\theta_0} [c_1 + 2c_2(\lambda^2 + 2J/\lambda - 3) \right. \\ &\quad \left. + 3c_3(\lambda^2 + 2J/\lambda - 3)^2] \right\}. \end{aligned} \quad (6.10)$$

We further specify thermal constants as follows:

$$\begin{aligned} \theta_0 &= 293.15 \text{ [K]} \quad (20^\circ \text{C}), \quad \gamma_0 = 22.333 \cdot 10^{-5} \text{ [1/K]}, \\ c_0 &= 2.2 \text{ [MPa/K]}. \end{aligned} \quad (6.11)$$

Fixing entropy in (6.10) it is possible to find the relationship between stretches and temperature. Such a relationship is shown graphically in Fig. 3. The entropy magnitude was set $\eta = 0.399314$ [KPa/K] to fit the room temperature to the reference state without stretches.

It is interesting to observe in Fig. 3 the appearance of the thermoelastic inversion point at $\lambda_{inv} = 1.062$ which was first discovered experimentally by Joule [10]. Though comparison of Joule’s results with the theoretical prediction is very

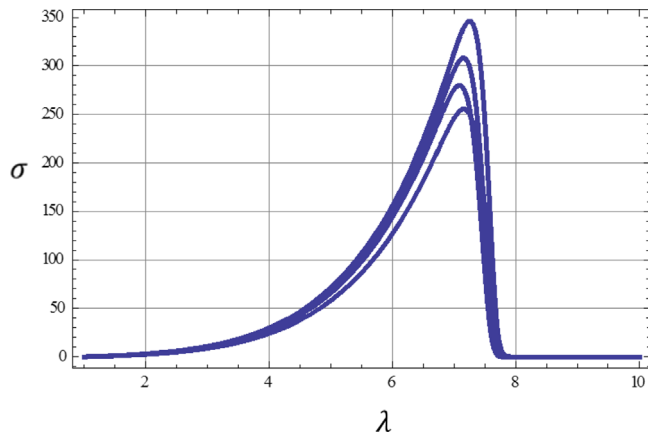


Fig. 4. Cauchy stress [MPa] versus stretch in uniaxial tension corresponding to temperatures 20 °C, 70 °C, 90 °C, 110 °C from the bottom to the top accordingly.

encouraging it should not be overstated. Indeed, to calibrate the theoretical model we used the experimental data for material from Hamdi et al. [8] while Joule, most likely, used a different material.

Now, fixing temperature in (6.8)–(6.9) it is possible to find the relationship between stresses and stretches. Such relationships are shown graphically in Fig. 4 for varying temperature.

It is readily seen in Fig. 4 that heating leads to material stiffening. This effect is well-known experimentally and its mechanism is related to rotations of bonds and molecular conformations at the microscopic level: Anthony et al. [2]; Treloar [20]; Chadwick [3]; Saccomandi and Ogden [17]. As a result of the stiffening, we observe the increase of the tensile strength – the limit points in Fig. 4. While the strength increases the corresponding critical stretch is almost constant. No experiments are known to us which consider strength under heating. It is reasonable to assume, nevertheless, that the critical stretch related to the length of unfolded molecular chains should not be affected by heating. It is hoped that the mentioned predictions concerning the strength alterations will encourage and guide experiments.

7. Concluding remarks

In the present work, we developed a new formulation of the constitutive equations for finite thermoelasticity of rubberlike solids. The main feature of this formulation is an account of failure processes via introduction of energy limiters in the Helmholtz free energy function. The introduction of the energy limiters gives a natural way to bound the thermoelastic energy that a material volume can store and dissipate. The latter also automatically limits stresses that can develop at a material point and, therefore, suppresses unphysical singularities typical of the traditional formulations of solid mechanics.

We extended a sample constitutive model with failure (that was experimentally calibrated for both intact deformation and rupture at the room temperature) to the thermoelastic range. Considering the uniaxial tension problem with heating we found that the model can reproduce the experimentally observed phenomenon of the thermoelastic inversion point and also it reproduces stiffening under the temperature increase. Besides, we found that the stiffening leads to the increase of the tensile strength while the rupture stretch practically does not change. No experimental data is available (to the best of our knowledge) yet on the latter issue while this theoretical finding is, probably, intuitively reasonable. Indeed, the stiffening is a result of the

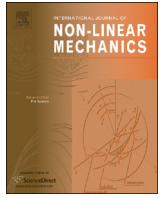
changing conformations of molecular chains that can be affected by heating. Rupture, on the other hand, is caused by the straightening of molecular chains whose unfolded length is not expected to alter under heating. Thus, the critical stretch of rupture does not change while the critical stress (strength) does.

We should note finally that the thermo-mechanical coupling might also have a regularizing effect on computer simulations of crack propagation where bulk failure tends to localize in thin bands. In the purely mechanical setting the failure may localize into zones whose thickness is dictated by the characteristic size of the spatial discretization – “mesh sensitivity” (e.g. [19]). Thus, the mesh size might affect the solution of a physical problem. That is undesirable, of course. In the case of thermoelastic coupling, however, the energy balance equation may create a regularization effect as long as heat flows. From the mathematical standpoint, the term with the divergence of the heat flux increases the order of the differential equations and allows for a solution of the boundary-layer type. Such boundary layer can naturally introduce the thickness of the localized damage zone (the crack). Importantly, the heat flux is still possible when the material failure process is already active. This circle of problems is very interesting yet requires numerical simulations that would go well beyond the scope of the present work.

References

- [1] J. Aboudi, Micromechanical analysis of the fully coupled finite thermoelastic response of rubber-like matrix composites, *Int. J. Solids Struct.* 39 (2002) 2587–2612.
- [2] R.L. Anthony, R.H. Caston, E. Guth, Equations of state for natural and synthetic rubberlike materials. I, *J. Phys. Chem.* 46 (1942) 826–840.
- [3] P. Chadwick, Thermo-mechanics of rubberlike materials, *Philos. Trans. R. Soc. A* 276 (1974) 371–403.
- [4] P. Chadwick, C.F.M. Creasy, Modified entropic elasticity of rubberlike materials, *J. Mech. Phys. Solids* 32 (1984) 337–357.
- [5] P. Chadwick, N.H. Scott, Linear dynamic stability in constrained thermoelasticity I. Deformation-temperature constraints, *Q. J. Mech. Appl. Math.* 45 (1992) 641–650.
- [6] A.N. Gent, P.B. Lindley, Internal rupture of bonded rubber cylinders in tension, *Proc. R. Soc. A* 2 (1959) 195–205.
- [7] M.E. Gurtin, E. Fried, L. Anand, *Mechanics and Thermodynamics of Continua*, CUP, 2010.
- [8] A. Hamdi, M. Nait Abdelaziz, N. Ait Hocine, P. Heuillet, N. Benseddiq, A fracture criterion of rubber-like materials under plane stress conditions, *Polym. Test.* 25 (2006) 994–1005.
- [9] G.A. Holzapfel, *Nonlinear Solid Mechanics*, Wiley, 2000.
- [10] J.P. Joule, On some thermo-dynamic properties of solids, *Philos. Trans. R. Soc. Lond.* A149 (1859) 91–131.
- [11] J. Kestin, *A Course in Thermodynamics*, 2 volumes, CRC Press, 1979.
- [12] I.-S. Liu, *Continuum Mechanics*, Springer, 2002.
- [13] G.A. Maugin, *The Thermomechanics of Nonlinear Irreversible Behaviours: An Introduction*, World Scientific, 1998.
- [14] K.N. Morman, A thermomechanical model for amorphous polymers in the glassy, transition, and rubber regions, AMD-Vol. 203, *Current Research in the Thermo-Mechanics of Polymers in the Rubbery-Glassy Range*, ASME, 1995.
- [15] I. Müller, W.H. Müller, *Fundamentals of Thermodynamics and Applications*, Springer, 2009.
- [16] R.W. Ogden, On the thermoelastic modeling of rubberlike solids, *J. Therm. Stress.* 15 (1992) 533–557.
- [17] G. Saccomandi, R.W. Ogden, *Mechanics and Thermomechanics of Rubberlike Solids*, Springer, 2004.
- [18] N.H. Scott, Linear dynamical stability in constrained thermoelasticity II. Deformation-entropy constraints, *Q. J. Mech. Appl. Math.* 45 (1992) 651–662.
- [19] P. Trapper, K.Y. Volokh, Modeling dynamic failure in rubber, *Int. J. Fract.* 162 (2010) 245–253.
- [20] L.R.G. Treloar, *The Physics of Rubber Elasticity*, OUP, 1975.
- [21] C. Truesdell, *The Tragicomical History of Thermodynamics, 1822–1854*, Springer, 1980.
- [22] C. Truesdell, *Rational Thermodynamics*, Springer, 1984.
- [23] K.Y. Volokh, Nonlinear elasticity for modeling fracture of isotropic brittle solids, *J. Appl. Mech.* 71 (2004) 141–143.
- [24] K.Y. Volokh, Hyperelasticity with softening for modeling materials failure, *J. Mech. Phys. Solids* 55 (2007) 2237–2264.
- [25] K.Y. Volokh, On modeling failure of rubber-like materials, *Mech. Res. Commun.* 37 (2010) 684–689.
- [26] K.Y. Volokh, Cavitation instability in rubber, *Int. J. Appl. Mech.* 3 (2011) 29311.

- [27] K.Y. Volokh, Review of the energy limiters approach to modeling failure of rubber, *Rubber Chem. Technol.* 86 (2013) 470–487.
- [28] K.Y. Volokh, On irreversibility and dissipation in hyperelasticity with softening, *J. Appl. Mech.* 86 (2014) 470–487.
- [29] K. Wilmanski, *Thermomechanics of Continua*, Springer, 1998.
- [30] O.H. Yeoh, Characterization of elastic properties of carbon black filled rubber vulcanizates, *Rubber Chem. Technol.* 63 (1990) 792–805.



Corrigendum

Corrigendum to “Nonlinear thermoelasticity with energy limiters”
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The sign of the second term on the right-hand side of Eq. (4.5) should be reversed:

$$\eta = -\frac{\partial\psi}{\partial\theta} - \frac{\partial g}{\partial\theta}\Pi. \quad (4.5)$$

The sign of the right-hand side of Eqs. (5.4), (5.6), (5.8) and (6.10) should be reversed and the second term in the braces should be multiplied by θ in Eq. (5.6):

$$\frac{\partial\psi}{\partial\theta} = H(\alpha)\exp\left(-\frac{W^m}{\Phi^m}\right)\left\{\frac{1}{\theta_0}\sum_{k=1}^3 c_k(I_1 - 3)^k - c_0 \ln \frac{\theta}{\theta_0}\right\}, \quad (5.4)$$

$$\frac{\partial^2\psi}{\partial\theta^2} = -H(\alpha)\exp\left(-\frac{W^m}{\Phi^m}\right)\frac{1}{\theta}\left\{c_0 + \frac{\theta m W^{m-1}}{\Phi^m}\left(\frac{1}{\theta_0}\sum_{k=1}^3 c_k(I_1 - 3)^k - c_0 \ln \frac{\theta}{\theta_0}\right)^2\right\}, \quad (5.6)$$

$$\eta = -H(\alpha)\exp\left(-\frac{W^m}{\Phi^m}\right)\left\{\frac{1}{\theta_0}\sum_{k=1}^3 c_k(I_1 - 3)^k - c_0 \ln \frac{\theta}{\theta_0}\right\} - \frac{\partial g}{\partial\theta}\Pi, \quad (5.8)$$

$$\begin{aligned} \eta = & -\exp\left(-\frac{W^m}{\Phi^m}\right)\left\{\frac{1}{\theta_0}\sum_{k=1}^3 c_k(\lambda^2 + 2J/\lambda - 3)^k - c_0 \ln \frac{\theta}{\theta_0}\right. \\ & + 3\gamma_0\exp[3\gamma_0(\theta - \theta_0)]\frac{2}{\lambda}\frac{\theta}{\theta_0}[c_1 + 2c_2(\lambda^2 + 2J/\lambda - 3) \\ & \left. + 3c_3(\lambda^2 + 2J/\lambda - 3)^2]\right\} \end{aligned} \quad (6.10)$$

The latter correction means that the graph in Fig. 3 is generated for the entropy $\eta = -0.399314$ [KPa/K].