On Cavitation in Rubberlike Materials

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Microscopic voids can irreversibly grow into the macroscopic ones under hydrostatic tension. To explain this phenomenon, it was suggested in the literature to use the asymptotic value of the hydrostatic tension in the plateau yieldlike region on the stress–stretch curve obtained for the neo-Hookean model. Such an explanation has two limitations: (a) it relies on analysis of only one material model and (b) the hyperelasticity theory is used for the explanation of the failure phenomenon. In view of the mentioned limitations, the objective of the present note is twofold. First, we simulate the cavity expansion in rubber by using various experimentally calibrated hyperelastic models in order to check whether the stress–stretch curves have the plateau yieldlike regions independently of the constitutive law. Second, we repeat simulations via these same models enhanced with a failure description. We find (and that was not reported in the literature) that the process of cavity expansion simulated via hyperelastic constitutive models exhibiting stiffening, due to unfolding of long molecules, is completely stable and there are no plateau yieldlike regions on the stress–stretch curves associated with cavitation. In addition, we find that the instability in the form of yielding observed in experiments does appear in all simulations when the constitutive laws incorporate failure description with energy limiters. [DOI: 10.1115/1.4032377]

Keywords: cavitation, soft materials, failure

1 Introduction

Many materials break as a result of growth and coalescence of small voids or cavities. For example, that is a scenario of ductile failure in metals. Rubberlike materials are also prone to growth of microcavities which may trigger macroscopic cracks. Gent and Lindley [1] performed "poker-chip" tests on natural rubber specimens and observed the yielding of micron-scale cavities into the visible ones. Globally, the specimen underwent uniaxial tension while locally the tension was hydrostatic because the specimen was thin and geometrically restrained.

To explain the experimental observations, Gent and Lindley [1] used the solution of the cavitation problem for neo-Hookean material which exhibited (a) plateau yieldlike region on the stretch–stress curve and (b) the (asymptotic) maximum magnitude of hydrostatic tension equal to 5/2 of the initial shear modulus or 5/6 of the initial elasticity (Young) modulus. Based on these notions, Gent [2–4] pushed forward the idea that the hydrostatic tension equal to 5/6 of the initial elasticity modulus was universally critical for cavitation instability in various rubberlike materials independently of the constitutive law. The latter idea is very attractive, of course, due to its simplicity. However, a closer look is required at its theoretical foundations based on the neo-Hookean material model. Indeed, the mentioned constitutive model derived from the restrictive statistical assumptions [5] is applicable to moderate stretches not exceeding the range from 1.2 to 1.5. At the same time, the asymptotic value of the critical hydrostatic tension is reached for stretches which are far beyond the range of the model applicability. Besides, the hyperelasticity theory was used for the explanation of the obviously inelastic failure phenomenon. Some controversy is evident.

Under the circumstances described above, it is natural: (a) to examine the solution of the cavitation problem for material models whose range of applicability goes well beyond the simplest neo-Hookean theory with the valid stretch range from 1.2 to 1.5 and (b) to simulate the hyperelastic material models enhanced with a failure description.

Particularly, in the present work, we simulate the cavity expansion via seven experimentally calibrated hyperelastic constitutive models reviewed by Marckmann and Verron [6]. In addition, we enhance these constitutive models with energy limiters to describe material failure and again analyze cavitation problem via the enhanced constitutive models.

The results of the simulations show that the process of cavity expansion described by hyperelastic constitutive models exhibiting stiffening due to unfolding of long molecules is completely stable, i.e., without the yieldlike plateau region and the critical asymptotic hydrostatic tension. However, the instability in the form of yielding, observed in experiments, does appear in simulations when the constitutive equations incorporate the material failure description.

The paper is organized as follows: The method of analysis is described in Sec. 2. The approach of energy limiters which allows a material failure description is briefly reviewed in Sec. 3. Various material models and results of the cavitation analysis for them are given in Sec. 4, and a brief discussion of the results in Sec. 5 completes the work.

2 Cavitation Analysis

Theoretical considerations of the expansion of bulk cavity in various materials can be found in Refs. [7–18]. Below, we follow the work [18].

Assuming that the deformation is centrally symmetric and the natural base vectors in spherical coordinates coincide with the principal directions of stretches, we can write the deformation law as follows:

$$r = r(R), \quad \vartheta = \Theta, \quad \omega = \Omega \tag{1}$$

where a material particle occupying position (R, Θ, Ω) in the initial configuration is moving to position (r, ϑ, ω) in the current configuration.

Designating the radial direction with index 1 and tangential direction with indices 2 and 3, we can write the principal stretches in the form

$$\lambda_1 = \frac{dr}{dR}, \quad \lambda_2 = \lambda_3 = \frac{r}{R} \tag{2}$$

Since the volume of incompressible material is preserved during deformation, we have

$$b^3 - a^3 = B^3 - A^3 \tag{3}$$

where A and a are the internal and B and b are the external radii of the sphere before and after deformation accordingly. We also note that any subsphere with the internal or external radius $r(R)$ should also preserve its volume and, consequently, we get

$$r^3 - a^3 = R^3 - A^3 \tag{4}$$

The principal components of the Cauchy stress are in the directions of the natural base vectors

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$$\sigma_1 = \sigma_{rr} = -p + \lambda_1 \frac{\partial \psi}{\partial \lambda_1}$$
$$\sigma_2 = \sigma_{\vartheta\vartheta} = -p + \lambda_2 \frac{\partial \psi}{\partial \lambda_2} \tag{5}$$
$$\sigma_3 = \sigma_{\omega\omega} = -p + \lambda_3 \frac{\partial \psi}{\partial \lambda_3}$$

where p is the indefinite Lagrange multiplier.

The stresses should obey the only equilibrium equation

$$\frac{d\sigma_{rr}}{dr} + 2\frac{\sigma_{rr} - \sigma_{\vartheta\vartheta}}{r} = 0 \tag{6}$$

This equation can be integrated as follows:

$$\sigma_{rr}(b) - \sigma_{rr}(a) = 2\int_a^b \frac{\sigma_{\vartheta\vartheta} - \sigma_{rr}}{r} dr \tag{7}$$

or

$$g = 2\int_a^b \left(\lambda_2 \frac{\partial \psi}{\partial \lambda_2} - \lambda_1 \frac{\partial \psi}{\partial \lambda_1}\right) \frac{dr}{r} \tag{8}$$

where boundary conditions have been taken into account

$$\sigma_{rr}(r = a) = 0, \quad \sigma_{rr}(r = b) = g \tag{9}$$

We note that the hydrostatic tension g is a function of the placement of the internal boundary, a, with account of

$$R(r, a) = \sqrt[3]{r^3 - a^3 + A^3} \tag{10}$$

For $b \gg a$, we have the problem of the expansion of small cavity in the infinite medium under the remote hydrostatic tension (In computations, we assume: $b = 1000$ A).

Remark 1. The analysis presented above is based on the assumption of central symmetry. We emphasize that such assumption correlates well with the experimental data. Indeed, the experimental data presented in Ref. [1] show that the visible voids remain spherical after the growth and unloading. No pronounced localization of failure into cracks is observed.

3 Hyperelasticity With Energy Limiters

A variant of the continuum description of bulk failure—*softening hyperelasticity* or *elasticity with energy limiters*—was developed in Refs. [19–24]. Softening hyperelasticity is dramatically simpler in formulation than any existing approach for modeling material failure: its basic idea is to introduce an energy limiter in the expression for strain energy. Such limiter enforces saturation—*the failure energy*—in the strain energy function, which indicates the maximum amount of energy that can be stored and dissipated by an infinitesimal material volume during rupture. The limiter induces stress bounds in the constitutive equations *automatically*.

The strain energy function for hyperelastic material with softening can be written in the following general form:

$$\psi = \psi^f - H(\alpha)\,\psi^e(\mathbf{C}) \tag{11}$$

$$\psi^f = \psi^e(\mathbf{1}) \quad \text{and}$$
$$\psi^e(\mathbf{C}) \to 0, \ \text{when} \ \|\mathbf{C}\| \to \infty \tag{12}$$

where ψ^f and $\psi^e(\mathbf{C})$ designate a constant bulk *failure* energy and an *elastic* energy, respectively; $H(z)$ is a unit step function, i.e., $H(z) = 0$ if $z < 0$ and $H(z) = 1$ otherwise; $\mathbf{1}$ is a second-order identity tensor; $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is the right Cauchy–Green tensor; $\mathbf{F} = \mathrm{Grad}\,\mathbf{y}$ is the deformation gradient, where $\mathbf{y}$ is the current placement of a material point which occupied position $\mathbf{x}$ in a reference configuration; and $\|\mathbf{C}\| = \mathrm{tr}\mathbf{C}$, for example.

The switch parameter $\alpha \in (-\infty, 0]$ is defined by the evolution equation

$$\dot{\alpha} = -H(\varepsilon - \psi^e/\psi^f), \quad \alpha(t = 0) = 0 \tag{13}$$

where $0 < \varepsilon \ll 1$ is a dimensionless tolerance constant.

The physical interpretation of Eqs. (11) and (12) is straight: material response is hyperelastic as long as the stored energy is below its limit, ψ^f. When the latter limit is reached, then the stored energy remains constant for the rest of the deformation process, thereby making material healing impossible. Parameter α works as a switch: if $\alpha = 0$, the process is hyperelastic and reversible, and if $\alpha < 0$, the material is irreversibly damaged and the stored energy is dissipated. We emphasize that the switch parameter is not an internal variable like in damage mechanics.

Using the dissipation inequality, it is possible to derive constitutive law in the following form [21]:

$$\boldsymbol{\sigma} = -\frac{2H(\alpha)}{J}\mathbf{F}\frac{\partial \psi^e}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} \tag{14}$$

where $J = \det\mathbf{F}$.

The elastic energy can be defined as follows [21], for example:

$$\psi^e = \frac{\Phi}{m}\Gamma\left(\frac{1}{m}, \frac{W^m}{\Phi^m}\right) \tag{15}$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t}\,dt$ is the upper incomplete gamma function; $W(\mathbf{C})$ is the strain energy of *intact* material, i.e., without failure; Φ is the energy limiter, which can be calibrated in macroscopic experiments; and m is a dimensionless material parameter, which controls sharpness of the transition to material instability on the stress–strain curve. Increasing or decreasing m it is possible to simulate more or less steep ruptures of the internal bonds accordingly.

Substitution of Eq. (15) in Eq. (14) yields

$$\boldsymbol{\sigma} = \frac{2H(\alpha)}{J}\exp\left(-\frac{W^m}{\Phi^m}\right)\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} \tag{16}$$

We note, finally, that the account of dissipation via step function in Eq. (11) is necessary when the material unloading is sound as in the case of crack propagation, for example. Otherwise, the step function can be dropped from equations as in the subsequent analysis of cavity expansion.

4 Results

In what follows, we use the seven experimentally calibrated hyperelastic models of rubber taken from Marckmann and Verron [6]. The reader is advised to refer the mentioned review for details.

In order to calibrate the energy limiters we assume, following experimental observations, that the material rupture is abrupt and $m = 10$; and the energy limiter is fitted to the critical rupture stretch $\lambda_{\mathrm{cr}} = 7.0$ in uniaxial tension for all cases. This critical stretch corresponds to the experimental data for the natural rubber vulcanizate [25].

We start with the *neo-Hookean model* for the intact material

$$W = \frac{\mu}{2}(I_1 - 3) \tag{17}$$

$$\mu = 0.4\,\mathrm{MPa} \tag{18}$$

and the results of analysis are presented in Fig. 1.

The uniaxial tension curve for this material model without (dashed) and with (solid) energy limiters is presented on the left in Fig. 1. The hydrostatic tension versus hoop stretch curves are presented on the right in Fig. 1 accordingly.

The *Biderman model* for the intact material is defined as follows:

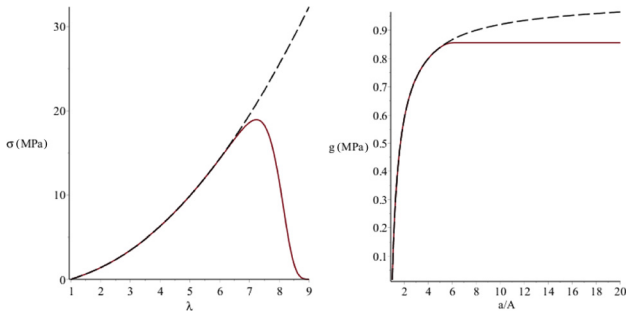

Fig. 1 Cauchy stress versus stretch in uniaxial tension (left) and hydrostatic tension versus hoop stretch in cavity expansion (right) for the neo-Hookean model

$$W = c_{10}(I_1 - 3) + c_{20}(I_1 - 3)^2 + c_{30}(I_1 - 3)^3 + c_{01}(I_2 - 3) \tag{19}$$

$$c_{10} = 0.208\,\text{MPa}, \quad c_{20} = -0.0024\,\text{MPa} \quad c_{30} = 0.0005\,\text{MPa}, \quad c_{01} = 0.0233\,\text{MPa} \tag{20}$$

and the results of analysis are presented in Fig. 2.

The *Yeoh–Fleming model* for the intact material is defined as follows:

$$W = \frac{a}{b}(I_m - 3)\left(1 - \exp\left(-\frac{I_1 - 3}{I_m - 3}b\right)\right) - c_{10}(I_m - 3)\ln\left(1 - \frac{I_1 - 3}{I_m - 3}\right) \tag{21}$$

$$a = 0.0519\,\text{MPa}, \quad b = 4.03, \quad I_m = 82.8, \quad c_{10} = 1.127\,\text{MPa} \tag{22}$$

and the results of analysis are presented in Fig. 2.

The *Isihara model* for the intact material is defined as follows:

$$W = c_{10}(I_1 - 3) + c_{20}(I_1 - 3)^2 + c_{01}(I_2 - 3) \tag{23}$$

$$c_{10} = 0.171\,\text{MPa}, \quad c_{20} = -0.00024\ \text{MPa}, \quad c_{01} = 0.00489\ \text{MPa} \tag{24}$$

and the results of analysis are presented in Fig. 2.

The *Ogden model* for the intact material is defined as follows:

$$W = \sum_{n=1}^{3} \frac{\mu_n}{\alpha_n}\left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right) \tag{25}$$

$$\mu_1 = 0.63\ \text{MPa}, \quad \alpha_1 = 1.3 \quad \mu_2 = 0.0012\ \text{MPa}, \quad \alpha_2 = 5 \quad \mu_3 = -0.01\ \text{MPa}, \quad \alpha_3 = -2 \tag{26}$$

and the results of analysis are presented in Fig. 3.

The *Gent model* for the intact material is defined as follows:

$$W = -\frac{E}{6}(I_m - 3)\ln\left(1 - \frac{I_1 - 3}{I_m - 3}\right) \tag{27}$$

$$E = 0.978\ \text{MPa}, \quad I_m = 96.4 \tag{28}$$

and the results of analysis are presented in Fig. 3.

The *Arruda–Boyce* (*8-chain*) *model* for the intact material is defined as follows:

$$W = \frac{nkT}{\sqrt{N}}\left\{\frac{1}{2}(I_1 - 3) + \frac{1}{20N}\left(I_1^2 - 9\right) + \frac{11}{1050N^2}\left(I_1^3 - 27\right) + \frac{19}{7000N^3}\left(I_1^4 - 81\right) + \frac{519}{673750N^4}\left(I_1^5 - 243\right)\right\} \tag{29}$$

$$nkT = 0.28\ \text{MPa}, \quad N = 25.4 \tag{30}$$

and the results of analysis are presented in Fig. 3.

Remark 2. Ball's formula [10] for the onset of cavitation can also be applied directly to all the strain energy functions. In the cases with energy limiters, in which the energy is bounded, the Ball formula provides results close to the yield stresses presented in our analyses above. In the cases of the stored energies without limiters, Ball's formula generally diverges. In addition, we should note that Ball [10] predicts the critical hydrostatic tension of the onset of cavitation in materials without pre-existing voids. The critical tension in the latter case is the bifurcation point after which the material behavior is not tracked. In the present analysis, on the contrary, it is assumed that the microscopic voids exist in advance and they expand under hydrostatic tension up to the critical yield point without any bifurcation.

5 Discussion

Experiments with rubberlike materials exhibit the possibility for microcavities to dramatically expand in hydrostatic tension—they become visible. Traditional explanation of this phenomenon relates the unstable expansion of voids—cavitation—to the plateau yieldlike regions on hoop stretch–hydrostatic tension curves observed for the hyperelastic neo-Hookean model. It was suggested that the asymptotic maximum value of the hydrostatic tension was the critical cavitation load. This load was equal to 5/2 of the initial shear modulus or 5/6 of the initial elasticity (Young)

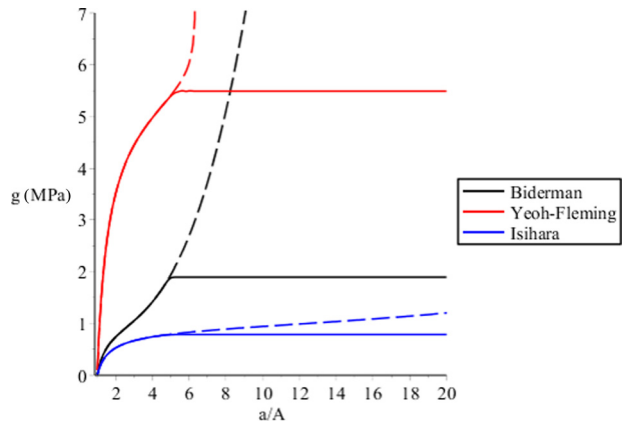

Fig. 2 Hydrostatic tension versus hoop stretch in cavity expansion for the Biderman, Yeoh–Fleming, and Isihara models

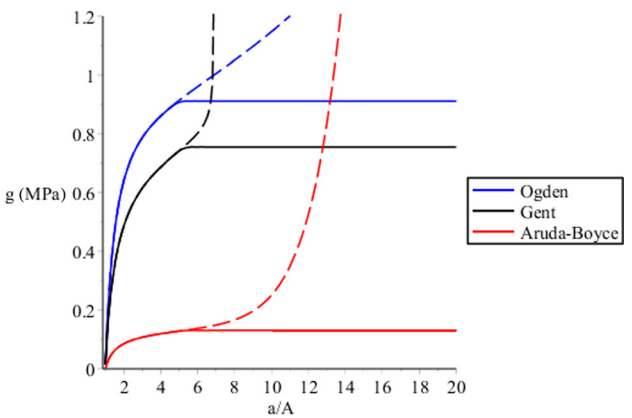

Fig. 3 Hydrostatic tension versus hoop stretch in cavity expansion for the Ogden, Gent, and Arruda–Boyce models

modulus. Moreover, it was suggested to use these critical numbers as universal indicators for cavitation in *any* rubberlike solid.

The weakness of the mentioned approach is twofold. First, the neo-Hookean model, which underlies it, is applicable to moderate stretches merely. Second, the hyperelastic theory cannot explain the obviously inelastic failure phenomenon in principle.

In order to address the mentioned points, we analyzed the cavity expansion problem by using seven experimentally calibrated hyperelastic constitutive models of rubber. The neo-Hookean model indeed showed yieldlike behavior in the cavitation problem. However, this model is generally inapplicable in the range of stretches where the yield starts because it cannot describe material stiffening observed in rubberlike solids. On the other hand, the models that describe material stiffening due to unfolding of long molecules—Biderman, Ogden, Gent, Yeoh–Fleming, Isihara, and Arruda–Boyce (8-chain)—did not show any yieldlike behavior in the problem of cavity expansion.

In addition, we enhanced all the mentioned hyperelastic constitutive models via material failure description by using the energy limiters. The limiters were introduced based on the assumption that materials fail in uniaxial tension at the stretch ratio equal to 7. Such an assumption is reasonable for the natural rubber vulcanizate. Simulations of the cavitation problem with the enhanced constitutive models showed clear yielding (due to material failure) for all the enhanced hyperelastic models including those with stiffening. Certainly, the cavity expansion gets unstable when the hydrostatic tension reaches a critical magnitude—no further load increase is necessary to provide the dramatic void growth.

We conclude that the cavitation problem is related to the damage of rubberlike solids and it can be predicted theoretically by using constitutive models enhanced with a failure description.

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