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Centenary of two pioneering theories in mechanics

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Abstract
This note discusses the Timoshenko–Ehrenfest beam theory and the Griffith fracture theory. Both were annunciated in the West in 1921, exactly a century ago. Much progress has been made in these fields. Discussing the deficiencies of the theories might pave way ahead.

Keywords
Elasticity, fracture, beam theory, Timoshenko, Griffith

1. On Timoshenko–Ehrenfest theory

The Timoshenko–Ehrenfest beam theory does not lack praise in the open literature. Laura et al. [1] maintained: "The publication, by Stephen Timoshenko, of his now classical theory of vibration of beams, whereby shear and rotatory inertia effects are taken into account, constitutes one of the most remarkable events in the development of the structural dynamics of the 20th century. Together with the Timoshenko–Mindlin theory of vibrating plates it has influenced the mathematical analysis of the quasi-infinite variety of dynamics of continuous media and structural acoustics problems from bridges and machine elements to surface, underwater and space vehicles passing through the prediction of the behavior to electronic packages, bioengineering systems etc." Archibald (see Howard [2]) characterized Timoshenko as "the patron saint of the American engineering". Frederick Terman (1900–1982), then-Provost-Emeritus of Stanford University, in his congratulatory letter to Timoshenko, in conjunction with the 90th anniversary of the birth of the latter, wrote,[1] on 2 December 1968: "I am pleased to report that the 'Timoshenko Legend' continues to flourish undiminished on the Stanford campus." According to Bhaskar [3], "the impact of Stephen Timoshenko's work in the area is undisputed (over a thousand citations in the last 25 years). His seminal paper [4] effected a major advancement to the theory following works of Euler, Bernoulli and Rayleigh…Timoshenko recognized the deficiency of the EB (Euler–Bernoulli) model and introduced a correction in his 1921 paper, now regarded as a classic in the field. The genius of his work lies in identifying shear of the cross section with respect to the axis as the most

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important degree of freedom missing in the EB model while still allowing that cross sections remain approximately plane during motion."

All probably would agree that the most important scientific contribution of Stephen Timoshenko to mechanics, as far as his research goes, is the beam theory. An overwhelming majority of authors provide as a reference his paper [4], not knowing that nearly the same paper was published by him a year prior to that [5]. Both papers were in English language, Timoshenko having left the Russian Empire. He had a derivation identical to his paper [5] in his Russian-language book on theory of elasticity [6]. In that book he recognized that the theory was developed together with Austrian-born Dutch scientist Paul Ehrenfest (1980–1933) when the latter lived in St. Petersburg. This fact of cooperation is recognized by Timoshenko in his English-language paper [7] as well as in the second edition of his Russian-language elasticity book [8], that appeared in the year of his death.

2. Inconsistency of Timoshenko–Ehrenfest Theory

The Timoshenko–Ehrenfest beam equations reads

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\left(1+\frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4}=0, \tag{1}$$

where E is modulus of elasticity, I is the moment of inertia, $w(x)$ is the beam displacement, ρ is the mass density, A is the cross-sectional area, k is the shear correction factor, G is the shear modulus, x is the axial coordinate, and t is the time.

Timoshenko (and Ehrenfest) [4–6] proceeds with an analysis of the beam that is simply supported at its both ends. Dividing (1) by ρA and using the notation for the squared radius of gyration, $r_g^2=I/A$, one obtains

$$a^2\frac{\partial^4 w}{\partial x^4}+\frac{\partial^2 w}{\partial t^2}-r_g^2\left(1+\frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}+r_g^2\frac{\rho}{kG}\frac{\partial^4 w}{\partial t^4}=0,\quad a^2=\frac{EI}{A}. \tag{2}$$

Setting

$$w(x,t)=\sin\frac{m\pi x}{L}(A_m\cos\omega_m t+B_m\sin\omega_m t) \tag{3}$$

satisfies the boundary conditions at the ends of the beam.

In (3), m denotes the number of half-waves in axial direction. Substitution of (3) into (2) leads to the following expression for natural frequencies

$$a^2\frac{m^4\pi^4}{L^4}-\omega_m^2-\omega_m^2\frac{m^2\pi^2 r_g^2}{L^2}-\omega_m^2\frac{m^2\pi^2 r_g^2}{L^2}\frac{E}{kG}+\frac{r_g^2\rho}{kG}\omega_m^4=0. \tag{4}$$

If one retains only first two terms in the frequency equation one obtains the following expression for the squared natural frequency

$$\omega_{m_{(0)}}=a\frac{m^2\pi^2}{L^2}=\frac{a\pi^2}{\lambda_m^2}, \tag{5}$$

where $\lambda_m=L/m$ is the length of half waves into which the beam is subdivided during vibration. Equation (5) coincides with the expression for the natural frequency of the classical, Bernoulli–Euler beam. With the above notation, Equation (4) is rewritten as

$$a^2\frac{\pi^4}{\lambda_m^4}-\omega_m^2-\omega_m^2\frac{\pi^2 r_g^2}{\lambda_m^2}-\omega_m^2\frac{\pi^2 r_g^2}{\lambda_m^2}\frac{E}{kG}+\frac{r_g^2\rho}{kG}\omega_m^4=0 \tag{6}$$

By retaining in (6) first three terms, one obtains

$$\omega_m = \frac{a\pi^2}{\lambda_m^2}\frac{1}{\sqrt{1+(\pi r_g \lambda_m)^2}}, \tag{7}$$

which coalesces with the expression produced for a Bresse–Rayleigh beam [9].

Timoshenko [4, 5] notes: "In order to obtain the *effect of shearing deformations*, we should take all of the terms in [Equation (6)] into consideration." The solution of the bi-quadratic equation for ω_m^2 in (6) leads to the following expression for squared natural frequency

$$\omega_m^2 = \frac{kG}{2r_g^2\rho}\left\{\left(1+\frac{\pi^2 r_g^2}{\lambda_m^2}+\frac{\pi^2 r_g^2}{\lambda_m^2}\frac{E}{kG}\right) \pm \left[\left(1+\frac{\pi^2 r_g^2}{\lambda_m^2}+\frac{\pi^2 r_g^2}{\lambda_m^2}\frac{E}{kG}\right)^2 - 4a^2\frac{\pi^4}{\lambda_m^4}\frac{r_g^2\rho}{kG}\right]^{\frac{1}{2}}\right\}.$$

As we see, we arrive at two branches of natural frequencies, namely the lower-frequency branch, denoted by subscript (1)

$$\omega_{m(1)}^2 = \frac{kG}{2r_g^2\rho}\left\{\left(1+\frac{\pi^2 r_g^2}{\lambda_m^2}+\frac{\pi^2 r_g^2}{\lambda_m^2}\frac{E}{kG}\right)^2 - \left[\left(1+\frac{\pi^2 r_g^2}{\lambda_m^2}+\frac{\pi^2 r_g^2}{\lambda_m^2}\frac{E}{kG}\right)^2 - 4a^2\frac{\pi^4}{\lambda_m^4}\frac{r_g^2\rho}{kG}\right]^{\frac{1}{2}}\right\}, \tag{8}$$

and higher-frequency branch, denoted by subscript (2)

$$\omega_{m(2)}^2 = \frac{kG}{2r_g^2\rho}\left\{\left(1+\frac{\pi^2 r_g^2}{\lambda_m^2}+\frac{\pi^2 r_g^2}{\lambda_m^2}\frac{E}{kG}\right)^2 + \left[\left(1+\frac{\pi^2 r_g^2}{\lambda_m^2}+\frac{\pi^2 r_g^2}{\lambda_m^2}\frac{E}{kG}\right)^2 - 4a^2\frac{\pi^4}{\lambda_m^4}\frac{r_g^2\rho}{kG}\right]^{\frac{1}{2}}\right\} \tag{9}$$

Timoshenko [4–6] did not pay attention to the two branches of frequencies. He never returned to this topic in his later books on vibrations either [10–14].

Instead, he examined the contribution of the last term in (6). The natural question arises: How can one ascertain the importance of the last term in (6) without first solving the equation (6)? Timoshenko [4–6] acts in an ingenious way. He states: "By substituting the first approximation [Equation (5)] for ω_m into the last term of this equation [i.e. Equation (1)], it can be shown that this term is a small quantity of the second order compared with the quantity: $\pi^2 r_g^2/\lambda_m^2$." Neglecting the last term in (6) reduces it to

$$a^2\frac{\pi^4}{\lambda_m^4} - \omega_m^2 - \omega_m^2\frac{\pi^2 r_g^2}{\lambda_m^2} - \omega_m^2\frac{\pi^2 r_g^2}{\lambda_m^2}\frac{E}{kG} = 0, \tag{10}$$

which leads to the following expression for the natural frequency

$$\omega_m = \frac{a\pi^2}{\lambda_m^2}\frac{1}{\sqrt{1+(\pi r_g/\lambda_m)^2(1+E/kG)}} \tag{11}$$

or, for the case when the following strong inequality holds,

$$\frac{\pi r_g}{\lambda_m}\left(1+\frac{E}{kG}\right) \ll 1. \tag{12}$$

One can use binomial expansion leading to approximate expression derived by Timoshenko [4–6]:

$$\omega_m = \frac{a\pi^2}{\lambda_m^2}\left[1-\frac{1}{2}\frac{\pi^2 r_g^2}{\lambda_m^2}\left(1+\frac{E}{kG}\right)\right]. \tag{13}$$

Timoshenko then considers a numerical example. He assumes $E=(8/3)G$ and takes a beam with $k=2/3$, resulting in

$$\frac{E}{kG}=4. \tag{14}$$

Timoshenko concludes: "Hence we see that the correction for shear is four times greater than the correction for rotatory inertia. The value of the correction of course increases with a decrease of the wavelength…, i.e. with an increase in *m*."

3. Cure for the deficiency in the Timoshenko–Ehrenfest equation

Based on this finding, Elishakoff and Lubliner [15] and Elishakoff and Livshits [16] omitted the last term and used the shorter equation

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\left(1+\frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}=0 \tag{15}$$

to derive several closed-form solutions for random vibrations of beams, under time-wise stationary and space-wise homogeneous white noise loading. At this juncture, let us pose a natural question: Does it make sense to attempt exact or closed-form solutions of approximate equations? The answer appears to be affirmative. Indeed, all textbooks list exact solutions of free and forced vibrations of uniform Bernoulli–Euler beams, which is cruder in some circumstances than (1), because both the shear deformation and rotary inertia are absent in the Bernoulli–Euler theory. Here one digression should be made to mention the paper [17], with the telling title "Timoshenko beam theory is not always more accurate than elementary beam theory."

At this juncture, it appears instructive to recall the rhetorical question posed by Novozhilov [18], "Let us ask, who is going to integrate approximately the equation of beam bending in the framework of the plane cross section hypothesis?"

Note that (15) does not contain the term $\left(\frac{\rho^2 I}{kG}\right)\partial^4 w/\partial t^4$ considered by Timoshenko. The latter, being a correction of the rotary inertia term, is obviously of higher order than the third term in (15). Hence, Equation (15) is both more consistent and simpler than the original Timoshenko–Ehrenfest equation (1), which contains all first-order correction terms but not all higher-order ones. Consistency demands that if one considers the last term in (1), other terms of the same order ought to be included as well.

Elishakoff [19] showed the way to obtain (15) directly, modifying the derivation by Timoshenko and Ehrenfest. It took nearly 100 years for this modification to be made after Timoshenko and Ehrenfest combined their heads in penning the refined theory of beams considering shear deformation. In the later study, Elishakoff et al. [20] showed that this equation is attainable via *asymptotic analysis* from elasticity equations. Thus, the complaint, made by van der Heijden et al. in [21], has been addressed. Elishakoff reports these developments and others in the recent monograph [9]. The further steps, beyond Timoshenko, were made in [22, 23] and others. For a general discussion of the need for refined theories, the interested reader can consult [24].

4. Griffith theory of fracture

Alan Arnold Griffith was the first to propose in 1921 a theoretical explanation of the fact that cracks can reduce load-bearing capacity of materials and structures [25]. He directly introduced a *pre-existing* crack into consideration and proposed a criterion of its instability: the onset of crack propagation. In particular, Griffith considered a plane with an elliptic hole and, using *global* energy balance, he derived the following formula for the critical hydrostatic tension

$$p_{cr}=\frac{K_{Ic}}{\sqrt{\pi(a^2-b^2)^{1/2}}}\sqrt{\frac{a^2-b^2}{a^2+b^2}}, \tag{16}$$

where K_{Ic} is a material constant called, in modern terms, mode I fracture toughness or the critical stress intensity factor (SIF); and a and b are the ellipse semi-axes along and orthogonal to the hole accordingly.

As $a \gg b$ for cracks, then we can rewrite (16) in a more familiar form

$$p_{cr} = \frac{K_{Ic}}{\sqrt{\pi a}}. \tag{17}$$

It is crucial that the critical tension is a function of the crack length whereas the crack sharpness, depending on the ratio b/a, is not involved in (17). This result is remarkable. Simply speaking, the Griffith criterion suggests that the crack sharpness does not affect the critical tension and, consequently, fracture.

If so, then it is possible to consider infinitely thin "mathematical cracks" with infinitely sharp (zero curvature) tips. That was the direction in which the main generalization of the Griffith theory called linear elastic fracture mechanics (LEFM) was developed. Expectedly, the infinitely sharp "mathematical cracks" led to the appearance of infinities (singularities) in stresses at the crack tip. Above all, the concept of the SIFs was introduced. The best physical interpretation of the latter concept belongs to Hutchinson [26] who called SIFs "truly esoteric quantities." Stress infinities and "mathematical cracks" do not seem to be less esoteric than SIFs. Fortunately, the lack of physical grounds never distressed enthusiasts of the Griffith (and LEFM) theory, e.g., [27].

5. Griffith theory versus physical experiments

Equation (17) can be used for the calibration of material fracture toughness

$$K_{Ic} = p_{cr}\sqrt{\pi a}. \tag{18}$$

Indeed, it is possible to create an initial crack of length $2a$ in a specimen and load it until the tension reaches critical value p_{cr}. Substituting p_{cr} and a in (18), we can experimentally calibrate the fracture toughness.

According to the Griffith theory described previously, the tip of the crack cannot affect the results of the measurement making the whole procedure very robust. That is in theory. Reality is different. Experiments show strong dependence of the results on the sharpness of the crack tip (e.g., [28–35]).

Experimentalists know that the calibration of fracture toughness is tough. It is an art. One should be truly artistic to fit theory that cannot be fitted. To help the experimentalists in this Sisyphean task, standardized tests were invented. It would not be exaggeration to conclude that the Griffith theory directly contradicts physical experiments. But who cares about experiments? Come on!

6. Griffith theory versus computer experiments

The mischievous reader might suggest that the discrepancy between theory and physical experiment does not come from Griffith, but rather from less than perfect experimentalists. Can we make a clean experiment? Yes, it is called *in silico* or computer experiment.

It is possible to reproduce real experiments on a computer with ideal loading of an ideal specimen. However, to obtain the critical tension p_{cr}, we need to introduce a description of material failure in the constitutive law. The latter can be done, for example, by bounding the stored energy of elastic material [36]. Indeed, the bond energy of atoms is bounded, and the number of atoms is limited; consequently, the macroscopic stored energy should also be bounded. The bounded stored energy automatically implies bounded stresses and the limit points on the stress–strain curves indicating material strength. Thus, strength becomes part of the constitutive description of a material, which cannot bear loads unlimitedly. The critical hydrostatic tension, at which the cracked plate fails to provide static stability, indicates the critical Griffith stress.

The described computer experiments with the bounded Hooke's stored energy function were performed to find the critical tension with various crack tips for mode I [37] and mode II [38] cracks.

In addition, the bounded neo-Hookean stored energy was used to simulate mode I cracks with various tips under moderately large stretches in soft materials [39].

All computer experiments confirmed the conclusion of physical experiments: the crack sharpness was crucially important for the onset of the crack instability, in contrast to the Griffith theory.

7. Griffith theory and phase field theories

Griffith theory gives a criterion of material failure in the presence of a crack. This criterion is separate from stress analysis (in the strength-of-materials style). Griffith could only dream about the possibility of tracking the onset, propagation, branching, and arrest of cracks in analysis. The development of computers made such a dream reality. Nowadays, research into theoretical fracture mechanics focuses on the development of initial boundary value problems (IBVPs) and computational schemes of their solutions for analysis of crack propagation. Such analyses usually use the following two families of approaches. The first is the surface failure models, also called cohesive surface models, which started with the celebrated work by Barenblatt [40]. The second is the bulk failure models, also called continuum damage mechanics, which started from the seminal paper by Kachanov [41]. All these approaches have pros and cons and even their facile review would take us far beyond the scope of this note.[2] Nevertheless, we briefly touch on the so-called phase field approach, which, according to its inventors and followers, presents a direct generalization of the Griffith theory [43–45].

For the sake of simplicity and brevity, we consider small deformation theory for brittle elastic material in the general form. The main idea is in introducing a new dimensionless field variable $\zeta \in [0, 1]$ called the phase field. This variable has zero value for purely intact material and it equals unity for a crack. With this new variable, the constitutive law takes the following form

$$\boldsymbol{\sigma} = \alpha(\zeta) \frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \tag{19}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are stress and strain tensors, respectively, w is the stored energy, and $\alpha(\zeta)$ is a function of the phase field.

The momentum balance equations and proper boundary and initial conditions are augmented with the equation describing the phase field

$$l^2 \nabla^2 \zeta = \beta(l, \zeta, \boldsymbol{\varepsilon}), \tag{20}$$

with boundary condition $\nabla\zeta \cdot \mathbf{n} = 0$, in which $\mathbf{n}$ is a unit outward normal to the body.

The squared characteristic length, l, appears on the left-hand side of (20) to provide dimensional consistency of the equation. This length is a small regularization parameter at the highest spatial derivatives of the phase field. The latter feature provides solutions for the phase field in the form of a boundary layer. This layer represents a diffused crack of finite thickness controlled by parameter l. However, and that is the very heart of the phase field approach, function $\beta(l, \zeta, \boldsymbol{\varepsilon})$ on the right-hand side of (20) has certain structure providing convergence of the diffused crack to the Griffith mathematical crack under decreasing parameter l.

In summary, the augmented IBVP should allow tracking cracks automatically. These cracks are diffused for non-zero regularization parameter l, and they should converge to the Griffith crack for the vanishing parameter $l \to 0$.

Unfortunately, and similar to the original Griffith theory, there are discrepancies between physics and mathematics in the phase field approach. Leaving aside the lack of the physical meaning of the phase field variable and equation describing it, we focus on the material failure process prior to the failure localization into crack. In the latter case, we have $\nabla\zeta = \mathbf{0}$ and (20) reduces to algebraic equation $\beta(l, \zeta, \boldsymbol{\varepsilon}) = 0$, which can be solved for the phase field

$$\zeta = \gamma(l, \boldsymbol{\varepsilon}). \tag{21}$$

Substitution of (21) in (19) yields

$$\boldsymbol{\sigma} = \eta(l, \boldsymbol{\varepsilon}) \frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \tag{22}$$

where $\eta(l, \boldsymbol{\varepsilon}) = \alpha(\gamma(l, \boldsymbol{\varepsilon}))$.

Constitutive equation (22) describes both deformation and failure of material prior to the failure localization. We note in passing that because material is elastic, a stored energy function ψ should exist obeying the condition $\partial\psi/\partial\boldsymbol{\varepsilon} = \eta(l, \boldsymbol{\varepsilon})\partial w/\partial\boldsymbol{\varepsilon}$.

Evidently, the varying regularization parameter l should not enter the physical description in the constitutive law while it does. The latter means that parameter l affects both material stiffness and strength and, consequently, it must be fixed for the given real material. The "regularization parameter" has physical value and it cannot therefore be varied.

Of course, we can exclude the characteristic length from function $\beta(\zeta, \boldsymbol{\varepsilon})$ in advance. Thus, we obtain a family of gradient damage theories, which do not describe Griffith cracks. In our humble opinion, that would be more physically appealing because real cracks should have finite thickness [46].

8. Appreciation of Timoshenko, Ehrenfest, and Griffith

There is apocryphal quote, attributed to physicist Werner Heisenberg, who once said that, if he were allowed to ask God two questions, they would be, "Why quantum mechanics? And why turbulence?" Supposedly, God would be able to answer the first question.

Turbulence is a great puzzle. Fracture is no less puzzling, yet it is of much more practical importance. The first step in solving a problem is always the most difficult: it gives direction, and it feeds critics. Griffith took the first step in fracture mechanics. He opened the gate.

Likewise, Timoshenko and his collaborator Ehrenfest opened the new research field of refined theories of beams, plates, and shells. These gentlemen deserve our appreciation even when we exercise our right of criticism.

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Notes

1. Terman, FE. *Letter to S.P. Timoshenko*, 2 December 1968 (available from the Bakhmeteff Archive of Russian and East European Culture, E.A. Vechorin Papers, Department of Rare Books and Manuscripts, Butler Library, Columbia University).
2. We mention, yet, that nonlocal continua formulations, e.g., peridynamics [42], are based (often tacitly) on the physical assumption of long-range particle interactions whereas the actual particle interactions are short-range, on the Ångstrom scale. Therefore, the physical basis for the nonlocal models appears disputable.

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