Tracking for nonlinear plants with multiple unknown time-varying state delays using sliding mode with adaptation

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SUMMARY

In this paper, we develop a sliding mode model reference adaptive control (MRAC) scheme for a class of nonlinear dynamic systems with multiple time-varying state delays which is robust with respect to unknown plant delays, to a nonlinear perturbation, and to an external disturbance with unknown bounds. An appropriate Lyapunov-Krasovskii type functional is introduced to design the adaptation algorithms, and to prove stability. Copyright © 2008 John Wiley & Sons, Ltd.

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INTRODUCTION

The variable structure control technique applied to uncertain systems with time-delays is a research area that is receiving considerable attention during the last few years, see e.g. [1]–[8] for the case of pure VSS problems, [9]–[13] for VSS combined with adaptive control methods, and the references in these papers. Mainly, robust stabilization problems were treated.

In [1] the problem of robust stabilization of a delayed linear systems with *known* delays in the presence of non-delayed nonlinear disturbances was studied. The cases with state feedback, static output feedback and dynamic output feedback were considered. An output feedback stabilizing controller for linear state-delayed systems with a *known* delay and with a nonlinear perturbation, norm-bounded by a linear function with *known* constants, was investigated in [2]. The paper [3] considers the sliding mode stabilization under state feedback for linear state delay plants with a disturbance bounded by a known function, and a known constant, or a time-varying state delay with a known bound. A sliding mode controller was developed by means of an LMI approach. With the use of LMI, and based on a polytopic nonlinear system formulation, the sliding mode state stabilization problem was investigated in [4] for a class of nonlinear state delay systems. The nonlinear perturbation was assumed to be bounded by a some known functional, and the cases considered were those of a known and constant delay, or a time-varying delay, bounded by a known constant.

The exponential state feedback sliding mode stabilization of linear systems with unknown bounded time-varying delays was considered in [6]. The design was based on the linear matrix inequalities (LMI) technique and a polytopic approach. Using the linear matrix inequalities technique, [5] considered the problem of robust state feedback sliding mode stabilization of uncertain time delay systems with structured unmatched system parameter uncertainties

and bounded external disturbances. LMI conditions were obtained, as well as a control law independent of the time delays.

In [7] the integral sliding mode technique was used for the design of a robust integral sliding mode compensator for optimal LQG control of linear stochastic systems with known input and observation delays. We would like to mention also [8] where output tracking was treated for nonlinear systems with an output time delay, replaced by its Padé approximation.

However, for the pure VSS case, the bounds of any uncertainty need to be available to the designer in advance. Knowledge of these bounds is a necessary condition for closed-loop stability. The relaxation of this shortcoming of the pure sliding control method motivates the combination of sliding mode robust control with adaptive approaches, whereby the knowledge of the upper bounds of the perturbations will not be required.

Only few combined adaptive—sliding mode results for delay systems have been published, see e.g. [9, 10, 11] in the centralized control case, and [12, 13] for decentralized control.

In [9] an adaptive state feedback robust stabilization problem is considered for a class of state delay systems with input nonlinearities. But also that paper restrictively assumes that the adaptive controller *knows* the bounds of the non-linear perturbation terms, and the bounds of the sector in which the input non-linearity resides. [10] deals with the state feedback stabilization of linear systems with *known* state delays, subject to bounded external disturbance with *unknown* bounds. The design guarantees convergence to a small ball. In [11] the variable structure MRAC problem is considered for a class of *stable* plants with a *known* input delay.

In [12] the problem of decentralized model reference adaptive variable structure control was investigated for a class of perturbed large-scale systems with time-varying time-delay in the

linear interconnections, and subject to local bounded perturbations, but under the assumptions that the factorization matrices in the matching conditions, as well as the upper bounds of the time delays are *known*. The sliding mode coordinated adaptive decentralized tracking for a class of nonlinear systems with state delays was considered in our recent paper [13], but also under the restrictive assumption that the controller *knows* the time delays.

In the present paper we introduce a new adaptive robust tracking scheme for a class of nonlinear dynamic system with multiple time-varying state delays. The proposed adaptive controller parametrization admits model reference adaptive designs with zero asymptotical errors without knowledge of the time delays, and with robustness properties with respect to state dependent delayed nonlinear perturbation, also in the presence of an unknown disturbance. Two cases of *a priori* knowledge about the state dependent delayed non-linearity are considered – the case when the non-linearity is bounded by some polynomial with unknown coefficients, and the case when the non-linearity is bounded by some known function of the state.

1. Plant model and problem formulation

We consider a class of nonlinear uncertain systems with state delays, suitably initialized, of the form

$$\dot{x}(t) = Ax(t) + bu(t) + bf\big(x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_M(t)), t\big) + bd(t) \tag{1}$$

where $x \in R^n$ is the state vector, $u(t) \in R$ is the control input, and $d(t) \in R$ is a bounded disturbance. The constant matrices $A \in R^{n \times n}$ and $b \in R^n$ have *unknown* elements. $f\big(x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_M(t)), t\big)$ is regarded as an *uncertain* state dependent nonlinear

perturbation. $\tau_j(t) \in R^+$, $j = 1, \ldots, M$ are *unknown* time-varying delays.

Our objective is to design a state feedback controller for Eq. (1) such that the closed-loop system is stable, and the states $x(t)$ asymptotically exact track the states of the non-delayed stable reference model

$$\dot{x}_r(t) = A_r x_r(t) + b_r r(t) \tag{2}$$

where $x_r(t) \in \mathbb{R}^n$ is the state vector, and $r \in \mathbb{R}$ is the reference input which is assumed to be a uniformly bounded and piecewise continuous function of time. The matrices A_r, b_r are known constant matrices of appropriate dimensions.

The following is assumed regarding the plant and the reference model: **(A1)** There exist an *unknown* constant vector $\theta_e^* \in R^n$ and a nonzero constant scalar θ_0 such that the following equations are satisfied,

$$A = A_r + b\theta_e^{*T}, \ b_r = b\theta_0. \tag{3}$$

(A2) The sign of θ_0 is known. **(A3)** The external disturbance $d(t)$ is bounded by an *unknown* constant $\|d(t)\| < d^*$.

For the nonlinear perturbation $f\big(x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_M(t)), t\big)$ two cases are considered. In case 1, we assume that **(A4)** the nonlinear function $f\big(x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_M(t)), t\big)$ is such that there exist nonnegative, but *unknown*, numbers ξ_{1l}^* and ξ_{2l}^* such that

$$|f(\cdot)| \leq \sum_{l=1}^{p} \xi_{1l}^* \|x(t)\|^l + \sum_{l=1}^{p} \sum_{j=1}^{M} \xi_{2l}^* \|x(t - \tau_j(t))\|^l$$

and that the order p of the polynomials is known; and, in case 2, we assume that **(A4')** there exists a continuous positive scalar valued function $\rho(x, t)$ and an *unknown* constant ρ^* such that

$$|f(\cdot)| \leq \rho^* \rho(x, t), \quad \text{for all } (t, x(t))$$

Moreover, for the time-varying delays $\tau_j(t)$, $j = 1, \ldots, M$ we assume that **(A5)** $\tau_j(t)$ is a nonnegative differentiable function, satisfying

$$0 \leq \tau_j(t) \leq \tau_{max}, \; \dot{\tau}_j(t) \leq \tau^* < 1 \tag{4}$$

where τ_{max} and τ^* are some *unknown* positive constants.

Remark 1. *The restrictive "matching condition" (A1) and and "sign assumption" (A2) are standard assumptions in the model reference control literature. Various techniques for their relaxation are known, see e.g. the textbooks [14, 15]*

2. Adaptive Controller 1

First we will develop the adaptive controller under the assumptions (A1–A4, A5). For the design of the adaptive controller, we incorporate model reference adaptive control (MRAC) methodology into the conventional two stage variable structure control (VSS) technique, see e.g. [16, 17].

First, it is necessary to design a sliding surface, like in conventional VSS design. Let us define the switching surface as follows:

$$S = G^T e = 0 \tag{5}$$

where $G \in R^n$ is some constant vector to be designed and $e(t) = x(t) - x_r(t)$ is the tracking error. If on a surface $S = 0$ there is a sliding mode, then G can be chosen such that the dynamics on the sliding surface has the desired closed-loop behavior, see e.g. [17].

There are several approaches to attaining a proper switching surface, see, e.g. the survey paper [19]. So we can use the procedure of the so-called equivalent control which is widely used in the variable structure control literature. The reader is referred to [17] for the details of the

method. During the sliding mode, the system satisfies the equations $S(e) = 0$ and $\dot{S}(e) = 0$ and

there exists a so-called equivalent control, such that error trajectories remain on the switching

manifold $S(e) = 0$. The equivalent control is found by recognizing that $\dot{S}(e) = 0$ is a necessary

condition for the error trajectory to remain on the switching manifold $S(e) = 0$. It can be

shown, e.g. [17], that in our case, by using (11) in $\dot{S}(e) = 0$, that the equivalent dynamic

equation for system in the sliding motion is of the form

$$\dot{e}_1(t) = (A_{r11} - A_{r12}G_2^{-1}G_1)e_1(t), \quad e_2(t) = -G_2^{-1}G_1^T e_1(t)$$

where $e = [e_1 \ e_2]^T$ is the block vector with $e_1 \in R^{n-1}, \quad e_2 \in R$ and the blocks $G_1, \quad G_2$ of

$G = [G_1^T, \ G_2]^T$ and $A_{r11}, \ A_{r12}, \ A_{r21}$ and A_{r22} of the matrix A_r have appropriate dimensions.

We note that in the sliding mode, the systems dynamics is independent of the disturbances

and insensitive to parameter uncertainties and we can easily assign the system performance

in the sliding mode just by selecting appropriate matrix G. If, on a surface $S = 0$, there is a

sliding mode, then the design parameters $G_1, \quad G_2$ can be chosen such that $\lim_{t \to \infty} e(t) = 0$

and the dynamics on the sliding surface has the desired closed-loop behavior.

The next step is to design an appropriate adaptive control law such that the sliding surface

will attract the error trajectories, and these trajectories will remain on the sliding surface for

all subsequent times. This step is described in following subsections.

2.1. Proposed control law

Under the assumption that conditions for the existence of a solution in the sense of Fillipov

are satisfied for the ensuing differential equations with discontinuous right hand side [20], we

look for a control law parametrization of the form

$$u(t) = \theta_e^T(t)e(t) + \theta_\omega(t)\omega(t) \tag{6}$$

where $\theta_e(t) \in \mathbb{R}^n$ and $\theta_\omega(t) \in \mathbb{R}$ are the vector and scalar adaptation gains, respectively. The signal $\omega(t) \in \mathbb{R}$ is defined as

$$\omega(t) = -Pe(t) - k_s \text{sgn}(S(t)) - k_0 \int_0^t \text{sgn}(S(t))dt - \sum_{l=1}^{p} k_l \|e(t)\|^l \int_0^t \text{sgn}(S(t)) \|e(t)\|^l \, dt \quad (7)$$

Here $P = G^T A_r + qG$ where $q > 0$ is to be chosen, the design parameters k_s, k_0 and $k_l, l = 1, \ldots, p$ are scalar positive constants, and $\text{sgn}(S(t)) = 1$, if $S(t) > 0$; $\text{sgn}(S(t)) = 0$, if $S(t) = 0$; $\text{sgn}(S(t)) = -1$, if $S(t) < 0$ and $\|\cdot\|$ denotes the Euclidian norm.

Remark 2. *The introduction of the signal $\omega(t)$ with its integral terms in (7), and its multiplication with the on-line updating scalar gain $\theta_\omega(t)$ in (6) is the main novel feature in our formulation of the adaptive control solution. Such a controller structure provides a convenient form of the error parametrization, see subsection 2.3 below, for the design of the adaptive controller in the presence of delayed nonlinearities with unknown delays and external disturbances.*

2.2. Adaptation algorithms

We now choose the adaptation algorithms as

$$\theta_e(t) = -z_1(0) - z_1(t) - z_1(t - h_1) - \int_0^t z_1(s)ds,$$

$$z_1(t) = \text{sgn}[\theta_0]\Gamma G^T b_r \text{sgn}(S(t))e(t)$$

$$\theta_\omega(t) = -z_2(0) - z_2(t) - z_2(t - h_2) - \int_0^t z_2(s)ds,$$

$$z_2(t) = \text{sgn}[\theta_0]\gamma G^T b_r \text{sgn}(S(t))\omega(t) \quad (8)$$

or in differential form

$$\dot{\theta}_e(t) = -z_1(t) - \dot{z}_1(t) - \dot{z}_1(t - h_1),$$

$$z_1(t) = \text{sgn}[\theta_0]\Gamma G^T b_r \text{sgn}(S(t))e(t)$$

$$\dot{\theta}_\omega(t) = -z_2(t) - \dot{z}_2(t) - \dot{z}_2(t - h_2),$$

$$z_2(t) = \text{sgn}[\theta_0]\gamma G^T b_r \text{sgn}(S(t))\omega(t) \tag{9}$$

where $\Gamma = \Gamma^T > 0$ and $\gamma > 0$ are some design matrix and scalar, respectively.

Remark 3. *For stability and exact asymptotic tracking, only the integral components $z_1(t)$ and $z_2(t)$ in (9) of the adaptation algorithm are needed, i.e. one may take $\dot{z}_1(t) = \dot{z}_2(t) = \dot{z}_1(t-h_1) = \dot{z}_2(t-z_2) = 0$. However, the use of the proportional $\dot{z}_1(t)$, $\dot{z}_2(t)$ and the proportional delayed $\dot{z}_1(t-h_1)$, $\dot{z}_2(t-h_2)$ terms in the adaptation algorithm (9) makes it possible to achieve better adaptation performance than with the traditional I and PI schemes [18]. Our adaptation algorithms include the traditional I and PI schemes as special cases. The design parameters h_1 and h_2 are chosen in the same way as the traditional gains Γ and γ in (8), (9).*

2.3. Error parametrization and stability analysis

The objective is now to find an adaptive control law such that the system trajectories are driven to the sliding surface. To develop such a law, it is necessary, as always in MRAC theory [14, 15], to express the closed-loop system in terms of the tracking error $e(t) = x(t) - x_r(t)$, and some parameter errors.

In view of (1), (2) and Assumptions (A1) and (A2) we obtain, after some manipulations,

$$\dot{e}(t) = A_r e(t) - b\theta_e^{*T} e(t) - b\theta_e^{*T} x_r(t) - b_r r(t) + bu(t)$$

$$+ bf\big(x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_M(t)), t\big) + bd(t) \tag{10}$$

Now, by adding and subtracting to the right part of (10) the term $b\theta_\omega^*\omega(t)$, and using (6) we have

$$\dot{e}(t) = A_r e(t) + b\tilde{\theta}_e^T(t)e(t) + b\tilde{\theta}_\omega(t)\omega(t) + b\theta_\omega^*\omega(t) - b_r r(t) - b\theta_e^{*T}x_r(t) + bf(\cdot) + bd(t) \quad (11)$$

where

$$\tilde{\theta}_e(t) = \theta_e(t) - \theta_e^* \quad \tilde{\theta}_\omega(t) = \theta_\omega(t) - \theta_\omega^* \quad (12)$$

are parameter errors. The unknown vector θ_e^* is from (3), and the unknown scalar θ_ω^* will be defined later.

In view of (A1), and by using (7) and (12), we obtain from (11) the following basic tracking error equation

$$\dot{e}(t) = A_r e(t) + b\tilde{\theta}_e^T(t)e(t) + b\tilde{\theta}_\omega(t)\omega(t) - b\theta_\omega^* P e(t) - b\theta_\omega^* k_s \mathrm{sgn}(S(t)) - b\theta_\omega^* k_0 \int_0^t \mathrm{sgn}(S(t))dt$$

$$- \sum_{l=1}^p b\theta_\omega^* k_l \|e(t)\|^l \int_0^t \mathrm{sgn}(S(t)) \|e(t)\|^l \, dt - b\theta_e^{*T} x_r(t) - b_r r(t) + bf(\cdot) + bd(t) \quad (13)$$

For stability analysis the following Lyapunov-Krasovskii functional is proposed

$$V = V_1 + V_2; \quad V_1 = |S(t)| + \sum_{j=1}^M \sum_{l=1}^p \int_{t-\tau_j(t)}^t \nu_{jl} \|e(t)\|^l$$

$$V_2 = \frac{1}{2|\theta_0|}\left(\tilde{z}_1^T(t)\Gamma^{-1}\tilde{z}_1(t) + \int_{t-h_1}^t z_1^T(s)\Gamma^{-1}z_1(s)ds\right) + \frac{1}{2|\theta_0|}\gamma^{-1}\left(\tilde{z}_2^2(t) + \int_{t-h_2}^t z_2^2(s)ds\right)$$

$$+ \frac{1}{2k_0}(\alpha(t) + \alpha^*\mathrm{sgn}(S(t)))^2 + \sum_{l=1}^p \frac{1}{2k_l}(\beta_l(t) + \beta_l^*\mathrm{sgn}(S(t)))^2 \quad (14)$$

where

$$\tilde{z}_1(t) = \tilde{\theta}_e(t) + z_1(t) + z_1(t - h_1), \quad \tilde{z}_2(t) = \tilde{\theta}_\omega(t) + z_2(t) + z_2(t - h_2). \quad (15)$$

The signals $\alpha(t)$ and $\beta_l(t)$, the parameters $\nu_{jl} > 0$, and the positive constants α^* and β_l^* $(l = 1, \ldots, p)$ will be defined later.

Remark 4. *In order to prove stability of the adaptive control scheme proposed here, some modifications with respect to the commonly used Lyapunov-Krasovskii functional were found necessary: we had to introduce auxiliary integral terms with the signals z_1 and z_2, and the terms with the "virtual" scalar adaptation gains $\alpha(t)$ and $\beta_l(t), l = 1, \ldots, p$.*

As it is known [16] global reaching condition is given by $\dot{V} < 0$ when $S(t) \neq 0$ and finite reaching time is guaranteed by $\dot{V} < -\epsilon$, $\epsilon > 0$ when $S(t) \neq 0$.

Assume that $G^T b_r \neq 0$, and define $\theta_\omega^* = \theta_0 (G^T b_r)^{-1}$. Then by using (3), (5) and P from (7), the time derivative of V_1 along (14) when $S(t) \neq 0$ can be written as

$$
\begin{aligned}
\dot{V}_1(t)|_{(13)} = & - q|S(t)| + \text{sgn}(S(t)) G^T b_r \theta_0^{-1} \tilde{\theta}_e^T(t) e(t) + \text{sgn}(S(t)) G^T b_r \theta_0^{-1} \tilde{\theta}_\omega(t) \omega(t) - k_s \\
& + \sum_{j=1}^{M} \sum_{l=1}^{p} \nu_{jl} (1 - \dot{\tau}_j(t)) \big[\|e(t)\|^l - \|e(t - \tau_j)\|^l \big] - k_0 \text{sgn}(S(t)) \int_0^t \text{sgn}(S(t)) dt \\
& - \sum_{l=1}^{p} k_l \text{sgn}(S(t)) \|e\|^l \int_0^t \text{sgn}(S(t)) \|e\|^l \, dt - \text{sgn}(S(t)) G^T b \theta_e^{*T} x_r(t) \qquad (16) \\
& - \text{sgn}(S(t)) G^T b_r r(t) + \text{sgn}(S(t)) G^T b f(x(t), x(t - \tau), t) + \text{sgn}(S(t)) G^T b d(t)
\end{aligned}
$$

Using the inequalities from (A4) and (A5) and boundedness of the reference signals ($|r(t)| \leq r^*, \|x_r\| \leq x_r^*$) we can write the following estimates for the last terms of (16)

$$
-\text{sgn}(S(t)) G^T b \theta_e^{*T} x_r(t) \leq |\text{sgn}(S(t))| \, \big\| G^T b \theta_e^{*T} \big\| \, \|x_r(t)\| \leq |\text{sgn}(S(t))| \, \eta_1 \qquad (17)
$$

$$
-\text{sgn}(S(t)) G^T b_r r(t) \leq |\text{sgn}(S(t))| \, \big| G^T b_r \big| \, |r(t)| \leq |\text{sgn}(S(t))| \, \eta_2 \qquad (18)
$$

$$
\text{sgn}(S(t)) G^T b d(t) \leq |\text{sgn}(S(t))| \, \big| G^T b \big| \, |d(t)| \leq |\text{sgn}(S(t))| \, \eta_3 \qquad (19)
$$

$$
\begin{aligned}
\text{sgn}(S(t)) G^T b f(\star) \leq & \sum_{j=1}^{M} \sum_{l=1}^{p} |\text{sgn}(S(t))| \, \bar{\xi}_{2jl} \, \|e(t - \tau_j)\|^l \\
& + \sum_{l=1}^{p} |\text{sgn}(S(t))| \, \bar{\xi}_{1l} \, \|e(t)\|^l + |\text{sgn}(S(t))| \, \eta_4 \qquad (20)
\end{aligned}
$$

where $\eta_1 = \left\| G^T b \theta_e^{*T} \right\| x_r^*$, $\eta_2 = \left| G^T b_r \right| r*$, $\eta_3 = \left| G^T b \right| d^*$, $\eta_4 = \sum_{j=1}^{M} \sum_{l=1}^{p} \left| G^T b \right| \left(\xi_{1jl}^* + \xi_{2jl}^* \right) x_r^*$

$\bar{\xi}_{1l} = \sum_{j=1}^{M} \left| G^T b \right| \xi_{1jl}^*$ and $\bar{\xi}_{2jl} = \left| G^T b \right| \xi_{2jl}^*$.

The time derivative of V_2 for $S(t) \neq 0$ satisfies

$$\dot{V}_2(t)|_{(13)} = \tfrac{1}{|\theta_0|} \tilde{z}_1^T(t) \Gamma^{-1} \dot{\tilde{z}}_1(t) + \tfrac{1}{|\theta_0|} \gamma^{-1} \tilde{z}_2(t) \dot{\tilde{z}}_2(t) + \tfrac{1}{2|\theta_0|} z_1^T(t) \Gamma^{-1} z_1(t) - \tfrac{1}{2|\theta_0|} \gamma^{-1} z_2^2(t - h_2)$$

$$- \tfrac{1}{2|\theta_0|} z_1^T(t - h_1) \Gamma^{-1} z_1(t - h_1) + \tfrac{1}{2|\theta_0|} \gamma^{-1} z_2^2(t) + \tfrac{1}{k_0} \big(\alpha(t) + \alpha^* \mathrm{sgn}(S(t)) \big) \dot{\alpha}(t)$$

$$+ \sum_{l=1}^{p_1} \tfrac{1}{k_l} \big(\beta_l(t) + \beta_l^* \mathrm{sgn}(S(t)) \big) \dot{\beta}_l(t) \tag{21}$$

By using of (9) it follows from (15) that

$$\dot{\tilde{z}}_1(t) = -z_1(t), \quad \dot{\tilde{z}}_2(t) = -z_2(t) \tag{22}$$

and

$$\tfrac{1}{|\theta_0|} \tilde{z}_1^T(t) \Gamma^{-1} \dot{\tilde{z}}_1(t) = -\mathrm{sgn}(S(t)) G^T b_r \theta_0^{-1} \tilde{\theta}_e^T(t) e(t) - \tfrac{1}{|\theta_0|} z_1^T(t) \Gamma^{-1} z_1(t)$$

$$- \tfrac{1}{|\theta_0|} z_1^T(t - h_1) \Gamma^{-1} z_1(t))$$

$$\tfrac{1}{|\theta_0|} \gamma^{-1} \tilde{z}_2(t) \dot{\tilde{z}}_2(t) = -\mathrm{sgn}(S(t)) G^T b_r \theta_0^{-1} \tilde{\theta}_\omega(t) - \tfrac{1}{\gamma|\theta_0|} z_1^2(t) - \tfrac{1}{\gamma|\theta_0|} z_2^2(t - h_2) \tag{23}$$

Applying (17)-(20) to (16), (23) to (21) and in view of the adaptation law from (9) one gets

$$\dot{V}(t)|_{(13)}^{S(t) \neq 0} \leq -q|S(t)| - k_s - \sum_{j=1}^{M} \sum_{l=1}^{p} \big(\nu_{jl} \bar{\tau} - \bar{\xi}_{2jl} \big) \| e(t - \tau_j(t)) \|^l - k_0 \mathrm{sgn}(S(t)) \int_0^t \mathrm{sgn}(S(t)) dt$$

$$- \sum_{l=1}^{p_1} k_l \mathrm{sgn}(S(t)) \| e \|^l \int_0^t \mathrm{sgn}(S(t)) \| e(t) \|^l \, dt + |\mathrm{sgn}(S(t))| \alpha^* + \sum_{l=1}^{p} \beta_l^* \| e(t) \|^l$$

$$+ \tfrac{1}{k_0} \big(\alpha(t) + \mathrm{sgn}(S(t)) \alpha^* \big) \dot{\alpha}(t) + \sum_{l=1}^{p} \tfrac{1}{k_l} \big(\beta_l(t) + \mathrm{sgn}(S(t)) \beta_l^* \big) \dot{\beta}_l(t) \tag{24}$$

$$- \tfrac{1}{2|\theta_0|} \big(z_1(t) + z_1(t - h_1) \big)^T \Gamma^{-1} \big(z_1(t) + z_1(t - h_1) \big) - \tfrac{1}{2|\theta_0|\gamma} \big(z_2(t) + z_2(t - h_2) \big)^2$$

where $\bar{\tau} = 1 - \tau^*$, $\alpha^* = \sum_{i=1}^{4} \eta_i$ and $\beta_l^* = \nu_l \bar{\tau} + \bar{\xi}_{1l}$.

From (24) we can conclude that a condition for the the third term to be negative-semidefinite

is that $\nu_{jl} > \bar{\xi}_{2jl}/\bar{\tau}$, $j = 1, \ldots, M$. Thus, because ν_{jl} in (14) are arbitrary positive parameters, we known that there exists some ν_{jl} such that whenever $\nu_{jl} > \bar{\xi}_{2jl}/\bar{\tau}$, $\left(\nu_{jl}\bar{\tau} - \bar{\xi}_{2jl}\right)$ is positive.

Let us define the "virtual" adaptation gains $\alpha(t)$ and $\beta(t)$ as

$$\dot{\alpha}(t) = - k_0 \text{sgn}(S(t)), \; \alpha(0) = 0$$

$$\dot{\beta}_l(t) = - k_l \text{sgn}(S(t)) \, \|e(t)\|^l \, , \; \beta(0) = 0, \; l = 1, \ldots, p \tag{25}$$

Then we obtain from (24)

$$\dot{V}(t)\big|_{(13)}^{S(t) \neq 0} \leq - q|S(t)| - k_s \tag{26}$$

Therefore $V(t)$ reaches the sliding surface $S(t) = 0$ in the finite time $t_r \leq V(0)/k_s$ and the signals $e(t)$, $\theta_e(t)$, $\theta_\omega(t)$ and $u(t)$ will be bounded before the system entering the sliding mode. According to the method of choosing G the error $e(t)$ will approach to zero as $t \to \infty$ when the closed system is in the sliding mode.

Remark 5. *Note that in light of the stability proofs, the control law* (6) *with the signal* $\omega(t)$ *from* (7) *and adaptation algorithms* (8), (9) *developed here do not depend on any delays.*

2.4. Main theorem for Adaptive Controller 1

Summarizing the above, we get

Theorem 1. *Consider system* (1) *and the reference model* (2)*. Suppose that assumptions (A1)-(A4), (A5) hold. Then the adaptive law* (6)-(7) *with update law* (8) *assures that the closed loop signals are bounded and that the tracking error* $e(t)$ *converges to zero asymptotically.*

3. Adaptive Controller 2

For this case (A4') of *a priori* knowledge about $f(\cdot)$, only the signal ω in (7) will be modified by simplification of the last term and by adding a special term. However, the structure of the control law (6) and the adaptation algorithms (8) will remain the same.

Remark 6. *This paragraph focuses on one of the simple ways to assignment bounding conditions. However, by following the path as before, the present design procedure may be easily extended to more relaxed version e.g. $f(.) \leq \sum_{k=0}^{p} \rho_k^* \rho_k(x,t)$, where ρ_k^* are unknown and $\rho_k(x,t)$ are known base functions.*

The signal $\omega(t)$ is modified to have the following form, where the first three terms are the same as in (7),

$$
\begin{aligned}
\omega(t) = & - Pe(t) - k_s \mathrm{sgn}(S(t)) - k_0 \int_0^t \mathrm{sgn}(S(t))dt \\
& - k_1 \left\| e(t) \right\| \int_0^t \mathrm{sgn}(S(t)) \left\| e(t) \right\| dt - k_2 \rho(x,t) \int_0^t \mathrm{sgn}(S(t))\rho(x,t)dt
\end{aligned}
\tag{27}
$$

The main result for Adaptive Controller 2 can be formulated as

Theorem 2. *Consider system* (1) *and the reference model* (2). *Suppose that assumptions (A1), (A2), (A3), (A4') and (A5) hold. Then the adaptive law* (6) *with ω form* (27) *and with the update law* (8) *assures that the closed loop signals are bounded and that the tracking error $e(t)$ converges to zero asymptotically.*

Proof. For the proof we will use a functional of the same type as (14) but with a modification of V_1 and V_2 as follows

$$
\begin{aligned}
V =&V_1 + V_2; \quad V_1 = |S(t)| + \sum_{j=1}^{M} \int_{t-\tau_j(t)}^{t} \nu_j \, \|e(t)\| \\
V_2 =&\tfrac{1}{2|\theta_0|}\Big(\tilde{z}_1^T(t)\Gamma^{-1}\tilde{z}_1(t) + \int_{t-h_1}^{t} z_1^T(s)\Gamma^{-1}z_1(s)ds\Big) + \tfrac{1}{2|\theta_0|}\gamma^{-1}\Big(\tilde{z}_2^2(t) + \int_{t-h_2}^{t} z_2^2(s)ds\Big) \\
&+ \tfrac{1}{2k_0}\big(\alpha(t) + \alpha^*\mathrm{sgn}(S(t))\big)^2 + \tfrac{1}{2k_1}\big(\beta(t) + \beta^*\mathrm{sgn}(S(t))\big)^2 + \tfrac{1}{2k_2}\big(\lambda(t) + \lambda^*\mathrm{sgn}(S(t))\big)^2
\end{aligned}
$$

$$(28)$$

where $\lambda(t)$ will be defined later.

Then in view of (27) and using (8) and (17)-(19) we can obtain the following estimate for the time derivative of V,

$$
\begin{aligned}
\dot{V}(t)|_{(13)}^{S(t)\neq 0} \leq &- q|S(t)| - k_s - k_0\mathrm{sgn}(S(t)) \int_{0}^{t} \mathrm{sgn}(S(t))dt \\
&- k_1\mathrm{sgn}(S(t)) \, \|e(t)\| \int_{0}^{t} \mathrm{sgn}(S(t)) \, \|e(t)\| \, dt + \mathrm{sgn}(S(t))G^Tbf(\cdot) \\
&- k_2\mathrm{sgn}(S(t))\rho(x,t) \int_{0}^{t} \mathrm{sgn}(S(t))\rho(x,t)dt + \alpha^* + \beta^* \, \|e(t)\| \\
&+ \tfrac{1}{k_0}\big(\alpha(t) + \alpha^*\mathrm{sgn}(S(t))\big)\dot{\alpha}(t) + \tfrac{1}{k_l}\big(\beta(t) + \beta^*\mathrm{sgn}(S(t))\big)\dot{\beta}(t) \\
&+ \tfrac{1}{k_2}\big(\lambda(t) + \lambda^*\mathrm{sgn}(S(t))\big)\dot{\lambda}(t)
\end{aligned}
$$

$$(29)$$

where $\alpha^* = \sum_{i=1}^{3} \eta_i$ and $\beta^* = \sum_{j=1}^{M} \nu_j$.

In view of (A4') we estimate the term $\mathrm{sgn}(S(t))G^Tbf(x(t), x(t-\tau), t)$ as

$$
\mathrm{sgn}(S(t))G^Tbf(\cdot) \leq \lambda^*\rho(x,t) \tag{30}
$$

where $\lambda^* = \big|\mathrm{sgn}(S(t))G^Tb\rho^*\big|$.

As above in (25), we now introduce the "virtual" adaptation gains

$$\dot{\alpha}(t) = -k_0 \text{sgn}(S(t)), \ \alpha(0) = 0$$

$$\dot{\beta}_l(t) = -k_1 \text{sgn}(S(t)) \|e(t)\|, \ \beta(0) = 0$$

$$\dot{\lambda}(t) = -k_2 \text{sgn}(S(t))\rho(x, t), \ \lambda(0) = 0 \tag{31}$$

Then by using (30) we obtain from (29)

$$\dot{V}(t)|_{(13)} \leq -q|S(t)| - k_s \tag{32}$$

Remark 7. *Note that selective parameters θ_ω^* from (12), ν_{il} from (14), α^*, β_l^* from (24), β^* from (29) and λ^* from (30) we use* only for analysis *and the controller does not use them.*

4. Simulation example

To illustrate the application of the proposed adaptive scheme, let us consider a plant defined by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -0.9 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0.9 & 0.75 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Big[u(t) + 0.1 sin(t)$$

$$+ \cos(x_1(t-\tau_1))x_2(t) + \sin(x_2(t-\tau_1))x_1(t) + \sin(x_2(t-\tau_4))x_2(t)$$

$$+ \cos(x_1(t-\tau_2))x_2(t-\tau_3) \Big], \quad x(0) = [-1 \ \ 1]^T \tag{33}$$

To build the adaptive controller we choose the reference model

$$\dot{x}_r(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_r(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} r(t), \quad x_r(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{34}$$

In this example all the parameters including $\tau_j, \ j = 1, \ldots, 4$ are unknown to the controller. The only information available to the controller is the structural information given in Assumptions A1– A5.

The controller (6), (8) is now found, based on the parameter values $\Gamma = 0.9\mathrm{I}, \quad \gamma = 0.45, \; k_s = 1, k0 = 0.2, k_1 = 0.1, h_1 = h_2 = 1, q = 0.01$ and $G^T = [1 \; 1]$.

The simulation results are shown in Figures 1 – 4, where we show the time responses of the plant state $x(t)$, the output of the reference model $x_r(t)$, the the input $r(t)$, the tracking error $e(t)$ and the deviation S from the switching surface $S = G^T e(t) = 0$. In the figures we included, for comparison, the time history of the signals also different sets of plant delays $\tau_j, \; j = 1, \ldots, 4$, namely $\tau_j = 9, 3, 5, 1$ and $\tau_j = 5, 4, 6, 9$, with the same controller (6), (8). From the graphs it is clear that the transients are only slightly sensitive to the delay variations. In particular, there are small differences in the responses at the times equalling the plant delays values, $\tau_j = 9, 3, 5, 1$ and $\tau_j = 5, 4, 6, 9$.
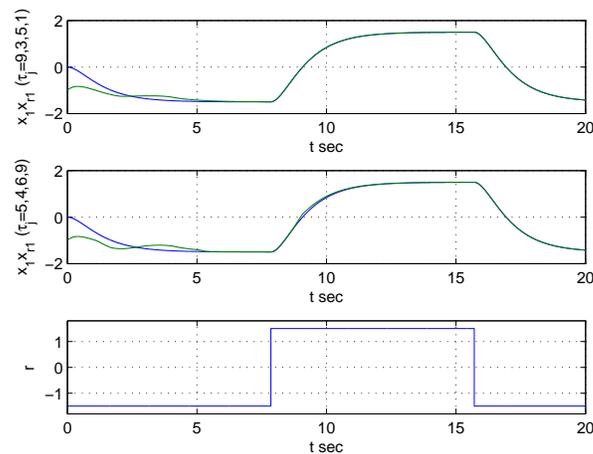

Figure 1. Simulation of the adaptive control system for the nonlinear plant with unknown state delays. The graphs show the time history of the first components of the signals $x(t) = [x_1(t), x_2(t)]^T$, $x_r(t) = [x_{r1}(t), x_{r2}(t)]^T$ and $r(t)$ for the two set of the plant delays, respectively.
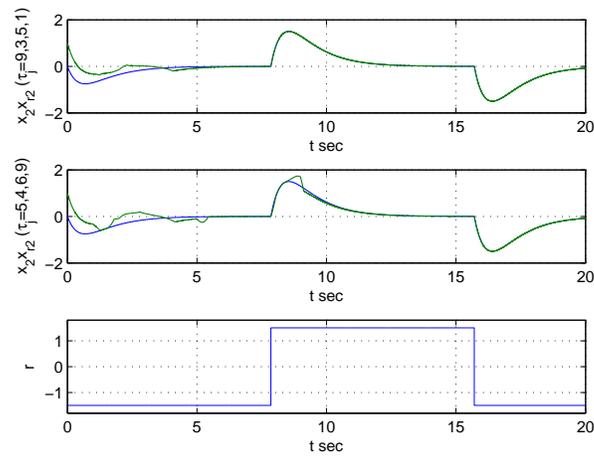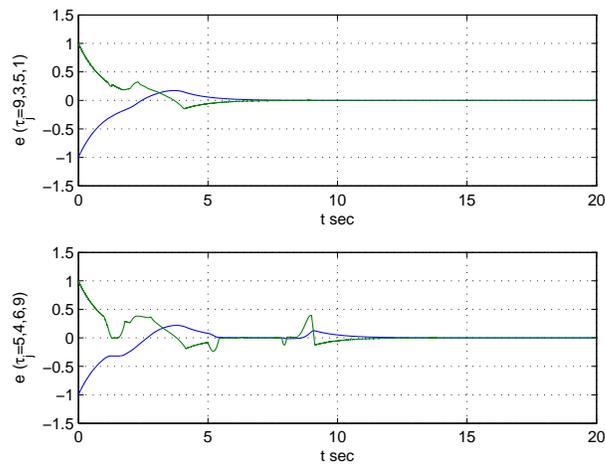
Figure 2. Simulation of the adaptive control system for the nonlinear plant with unknown state delays. The graphs show the time history of the second components of the signals $x(t) = [x_1(t), x_2(t)]^T$, $x_r(t) = [x_{r1}(t), x_{r2}(t)]^T$ and $r(t)$ for the two set of the plant delays, respectively.

Figure 3. Simulation of the adaptive control system for the nonlinear plant with unknown state delays. The graphs show the time history of the tracking error $e(t) = [e_1(t), e_2(t)]^T$ for the two set of the plant delays, respectively.

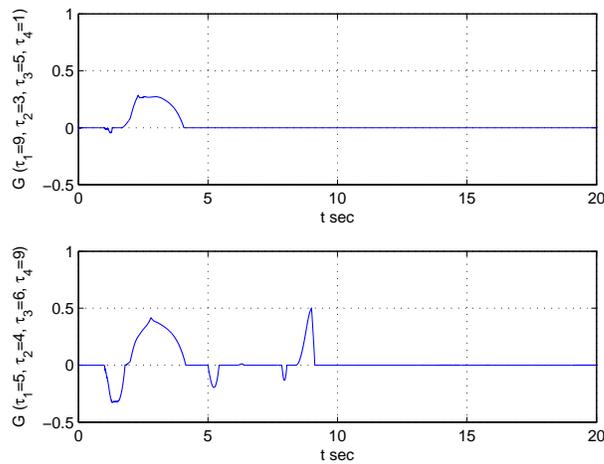*Int. J. Robust Nonlinear Control* 2008; **00**:1–20

Figure 4. Simulation of the adaptive control system for the nonlinear plant with unknown state delays. The graphs show the time history of the deviations S from the switching surface $S = G^T e(t) = 0$ for the two set of the plant delays, respectively.

5. Concluding Remarks

In this paper, two parameterizations for sliding mode model reference adaptive control have been proposed for a class of uncertain nonlinear state delay systems with unknown multiple time-varying delays and an external disturbance. It was shown that using novel adaptive controller parameterizations, together with standard variable system techniques, it is possible to design a state feedback controller that ensures the boundedness of the closed-loop signals, exact asymptotic tracking, and robustness with respect to two cases of nonlinear state dependent perturbations, an external disturbance, and time-varying state delays.

We propose to update the controller parameters with a proportional, integral time-delayed (PITD) adaptation algorithm. For a detailed motivation of this kind of adaptation, see [18]. A suitably selected Lyapunov-Krasovskii type functional with "virtual" adaptation gains is

proposed to design the update mechanism for the controller parameters, and to prove stability. Simulations demonstrate that the combined MRAC sliding mode controller has good tracking performance and robustness.

We believe that the new adaptive controller parametrization may be applied to various sliding mode adaptive tracking problems for plants with state delay, such as the case of multivariable systems and the case of output-feedback adaptive control. These cases are currently under investigation.

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