CROSS-OVER FREQUENCY LIMITATIONS IN MIMO NON-MINIMUM PHASE FEEDBACK SYSTEMS

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Abstract

This paper investigates limitations and design trade-offs of the closed-loop sensitivity/performance of linear-time-invariant non-minimum-phase uncertain MIMO plants, with l inputs and m outputs, where $m \leq l$. It is shown that if rows $i_1, ..., i_k$ of the plant transfer function form a $k \times l$ non-minimum phase transfer matrix, and if the design is such that the sensitivity gain of $k-1$ rows among the rows $i_1, ..., i_k$ of the closed-loop transfer function is low, then by necessity the sensitivity gain of the remaining row is high. This sensitivity constraint is quantified with the help of the cross-over frequency restriction of a specially constructed SISO transfer function that includes the right half plane zeros and poles of the $k \times l$ transfer matrix.

Index terms

Linear systems, sensitivity, multivariable systems, non-minimum phase, feedback control, bandwidth.

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1 Introduction

It is well known that the benefit of feedback for non-minimum-phase (NMP) plants, both SISO and MIMO, is limited. This NMP limitation appears when the plant has right-half-plane (RHP) zeros, pure delay or if the plant is sampled. Classic examples of NMP plants include flight control (aft δ_e control to elevation, and throttle command to elevation as measured close to the aircraft center of gravity), and the inverted pendulum.

The SISO case has been investigated widely in the literature. [25, 14] presented an optimal robust synthesis technique to design a feedback controller for an uncertain NMP plant to achieve a given closed-loop performance, providing the designer with insight into the trade-offs between closed-loop performance and bandwidth, and also defining an implicit criterion for determining whether a solution exists. [26] developed a criterion to estimate the maximum bandwidth of a sampled plant for given gain and phase margin, assuming an open-loop of the ideal Bode characteristics form and using asymptotic approximations. [12] extended this technique to stable plants with several RHP zeros, showing how to achieve a large open-loop gain in several frequency ranges, although there would always be some frequency ranges which are determined by the RHP zeros, in which the open-loop gain must be less than 0 dB. This known fact was proven in [6, 7] showing that for NMP plants, a small sensitivity in one frequency range forces a large sensitivity in the complementary range. [7, 8] developed several constraints on the closed-loop sensitivity of NMP and/or unstable plants in the form of weighted integrals of the sensitivity on all frequencies or on a frequency range where the open-loop gain is much less than 1. [21] used their results to provide a bandwidth limitation on NMP and/or unstable plants. For cross-over frequency limitations assuming a given slope of the open loop amplitude around the cross-over frequency, see [1].

The MIMO case is quite different from the SISO case: [5] showed that the RHP transmission zeros of a MIMO plant are also transmission zeros of the plant output in any closed-loop stable structure. [13] were the first to discuss the sensitivity of each element of the sensitivity transfer function of a MIMO plant showing that the MIMO quantitative feedback theory (QFT) design method can be applied to NMP plants where the cost is high sensitivity of at least all the elements of one row of the sensitivity transfer function, whereby the row can be chosen by the designer. This moving effect of the RHP zeros to a specific output was discussed in [27], Ch. 6.5. [30] showed explicitly the limitations of NMP plants in the LTR procedure. [22] developed performance limitations of NMP MIMO systems measured by the cheap quadratic functional. The main result is a quantitative measure for the degree of difficulty in solving the servomechanism problem for NMP systems which is related to $\sum 1/\lambda_i$ where λ_i is a RHP zero of the plant. [2] developed sensitivity integral relations by which the sensitivity trade-off in different frequency ranges as a function of the RHP poles and zeros were extended to continuous-time MIMO plants, and in [4] extended to discrete-time MIMO plants. For a multivariable system with RHP zeros, [3] developed for its singular values an integral relation, akin to Bode's phase-gain relation, as well as an integral sensitivity relation. [10] presented integral constraints, in the form of inequalities, for the sensitivity of unstable or non-minimum-phase MIMO feedback systems, giving insight into the sensitivity trade-offs and the cost of decoupling in multivariable design. For time domain interpretations, see [19]. The discrete-time multivariable case was also discussed in [11], where analytic constraints for the sensitivity and mixed sensitivity functions were given using coprime factorization and state space representation. The book [24] contains many of the integral results mentioned above for SISO and MIMO continuous-time and discrete-time plants.

Most results mentioned above are logarithmic integral constraints on the sensitivity and related functions. A different approach is taken in [1, 15]: constraints on the sensitivity elements and design trade-offs in feedback systems with RHP zeros and/or poles are expressed in form of a limitation on the achievable cross-over frequency. We follow this *cross-over frequency approach* in the belief that it is attractive from a practical point of view. In this paper, based on [28, 29], the cross-over frequency of a specially designed SISO transfer function that includes RHP zeros and poles from the MIMO plant, quantifies the sensitivity trade-off. [23] briefly suggests a similar but weaker condition, for which a proof is given in [16, 18].

1.1 Notation and assumptions

- A bold capital letter denotes a matrix and the same indexed letter denotes its entries.
- A lower case bold letter denotes a vector, and the same indexed lower case plain letter denotes its entries.
- The same letter denotes a time domain signal and its Laplace transform, with its arguments, t or s, explicitly shown only where an ambiguity may appear.
- $l(s) = l_M(s)l_A(s)$ denotes the factorization of $l(s)$ into its *minimum-phase stable factor*, $l_M(s)$, and all-pass factor, $l_A(s)$, whereby $l_M(s)$ is defined as the minimum-phase stable transfer function for which $|l_M(s)| = |l(s)|$.

Consider the feedback system shown in Fig. 1.

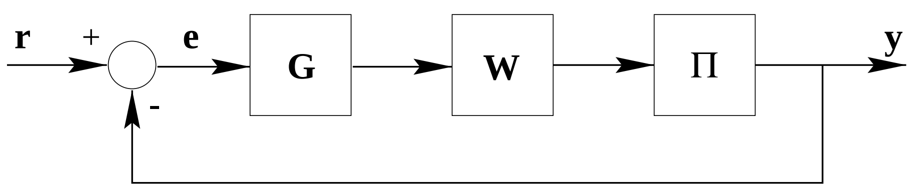

Figure 1: The MIMO feedback structure

- $\mathbf{\Pi}$ - an l input m output plant, $m \leq l$, of full rank.
- $\mathbf{G} = \text{diag}(g_i) = \text{diag}(\mathbf{G}_1, \mathbf{G}_2)$ is an $m \times m$ diagonal controller, where $\mathbf{G}_1$ is $r \times r$.

- $\mathbf{W}$ - a $l \times m$ full rank dynamic *weighting matrix*, such that $\mathbf{P} = \mathbf{\Pi W}$ is a square transfer matrix. In general, $\mathbf{W}$ is chosen prior to the design of $\mathbf{G}$ such that $\mathbf{P}$ can be seen as the plant when designing $\mathbf{G}$.
- $\Delta = \det(\mathbf{P})$, and Δ_{ij} is the ji minor of $\mathbf{P}$ multiplied by $(-1)^{i+j}$.
- $\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix}$, a partition of $\mathbf{P}$, $\mathbf{P}_{11}$ is $r \times r$.
- $\mathbf{Q} = \mathbf{P}^{-1} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}$, a partition of $\mathbf{Q}$, $\mathbf{Q}_{11}$ is $r \times r$.
- $\mathbf{P}^k$ is the matrix transfer function $\mathbf{P}$ whose k:th row and column have changed places with the first row and column, respectively, and similarly for $\mathbf{Q}^k$ and $\mathbf{G}^k$. The partitions $\mathbf{P}^k_{ij}$ and $\mathbf{Q}^k_{ij}$, $i = 1,2$, $j = 1,2$ are defined similarly to the partitions of $\mathbf{P}$ and $\mathbf{Q}$ above.
- $\mathbf{L} = \mathbf{\Pi WG} = \mathbf{PG}$, $m \times m$ open-loop transfer matrix.
- $\mathbf{S} = [\mathbf{I} + \mathbf{L}]^{-1} = [s_{ij}]$, the sensitivity matrix.

Throughout this paper we will assume: (i) $\mathbf{\Pi}$, $\mathbf{L}$, $\mathbf{WG}$ and $\mathbf{G}$ are strictly proper full rank matrix transfer functions of rational polynomials; (ii) there are no common RHP poles or zeros between $\mathbf{\Pi}$ and $\mathbf{W}$, $\mathbf{\Pi}$ and $\mathbf{G}$, or $\mathbf{W}$ and $\mathbf{G}$. We define (iii) RHP is the closed right half plane; and (iv) Rosenbrock's definition of RHP poles and zeros by the Smith-McMillan form [20] is used.

2 SISO Plants

In this section, some definitions and results for SISO systems are quoted as a background to the MIMO case. The proofs are found in [29].

Definition 2.1 *Let $\mathcal{L}_0$ be defined as the set of rational transfer functions without poles or zeros on the imaginary axis such that $l(s) \in \mathcal{L}_0$ iff $|l(0)| > 1$; $1+l(s)$ does not have RHP zeros; and there exists a single frequency, ω_c, for which $|l(j\omega_c)| = 1$ and $\arg(l(j\omega_c)) > -180°$. The frequency ω_c is called the* cross-over frequency*, and the* phase margin *equals* $\arg(l(j\omega_c))+180°$.

Lemma 2.1 *For any $l(s) \in \mathcal{L}_0$ such that $|l(j\omega)| < 1$ for all $\omega > \omega_c$, it follows that* $\arg(l_M(j\omega_c)) \leq 0$.

Definition 2.2 *Let $\mathcal{L}_1$ be the set of transfer functions such that $l(s) \in \mathcal{L}_1$ iff $l(s) \in \mathcal{L}_0$, and* $\arg(l_M(j\omega_c)) \leq 0$*, where l_M is the minimum-phase stable factor of $l(s)$.*

The main result for the SISO case is the upper and lower bounds on the cross-over frequency of an open-loop transfer function belonging to $\mathcal{L}_1$:

Lemma 2.2 *Let $l(s) \in \mathcal{L}_1$ be any open-loop transfer function of a closed-loop system, with phase margin Φ and cross-over frequency ω_c. Let $l(s) = l_M(s)l_A(s)$ be the factorization of $l(s)$ into its minimum-phase stable, $l_M(s)$, and all-pass, $l_A(s)$, factors. Suppose that there exist solutions to*

$$\Phi - \arg l_A(j\omega) = 180^\circ, \tag{1}$$

where ω_1 (ω_2) is the lowest (highest) frequency that solves (1) and $\omega_1 = \omega_2$ if there exists a single solution. Then

1. *$\omega_c \leq \omega_1$ if $l_A(s)$ has RHP zeros and is stable;*

2. *$\omega_c \geq \omega_2$ if $l_A(s)$ has no RHP zeros and is unstable; and*

3. *$\omega_1 \leq \omega_c \leq \omega_2$ if $l_A(s)$ has both RHP poles and zeros.*

Corollary 2.1 *It is clear from (1) that ω_1 and ω_2 of Lemma 2.2 are continuous functions of the RHP zeros and poles of the plant. Moreover, if an open-loop $l_1(s)$ includes the RHP poles and zeros of another open-loop $l_2(s)$, then the allowed cross-over frequency range of $l_2(s)$ includes that of $l_1(s)$.*

The sets $\mathcal{L}_0$ and $\mathcal{L}_1$ are discussed in [29], where it is demonstrated that many transfer functions of practical interest are covered. Similar qualitative results which give lower and upper limits on the bandwidth of a closed-loop system due to open-loop RHP poles and/or zeros are given in [21]. The significance of Lemma 2.2 is that an NMP factor in an open-loop transfer function "steals" phase in such a way that the cross-over frequency is restricted, and hence the frequency range over which sensitivity may be reduced.

3 MIMO Plants

The MIMO sensitivity function, $\mathbf{S} = [\mathbf{I} + \mathbf{L}]^{-1}$, is a natural extension of the SISO sensitivity function. As for SISO plants, it can be shown that in general, smaller values of $|s_{ij}|$ result in reduced closed-loop sensitivity to plant variations and better disturbance rejection. Thus smaller values of $|s_{ij}|$ over a larger frequency range are considered more desirable. Similarly to the SISO case we shall use the cross-over frequency of some open-loop transfer functions as a design tool to evaluate the constraints on $|s_{ij}|$. We will show the sensitivity reduction limitations on the elements of one row of $\mathbf{S}$, provided the sensitivities of all other rows tend to zero, and to which set of rows the high sensitivity row belongs. It will also be shown how many rows are sensitivity reduction limited.

3.1 Limitations for a single row of S

In this section we will show the sensitivity reduction limitations on the elements of one row of $\mathbf{S}$, provided the sensitivities of all other rows are small. To show this we will first establish that under the condition that all loops, except the kth, are closed with sufficiently large diagonal controller gains, $|g_i(s)|$, $i \neq k$, the cross-over frequency of the kth loop is restricted, according to Lemma 2.2, by the RHP zeros of $\det(\mathbf{P})/\Delta_{kk}$, where Δ_{kk} is the signed kk minor of $\mathbf{P}$.

Referring to Section 1.1, let a partition of $\mathbf{P}^k$ be

$$\mathbf{P}^k = \begin{bmatrix} \mathbf{P}^k_{11} & \mathbf{P}^k_{12} \\ \mathbf{P}^k_{21} & \mathbf{P}^k_{22} \end{bmatrix}$$

where $\mathbf{P}^k_{11}$ is 1×1. Let Δ_{kk} denote the signed kk minor of $\mathbf{P}$ = the 11 minor of $\mathbf{P}^k$, $\Delta = \det(\mathbf{P}) = \det(\mathbf{P}^k)$, and $\mathbf{M}^k$ the matrix transfer function

$$\mathbf{M}^k = \begin{bmatrix} \mathbf{I} + \mathbf{P}^k_{11} g_k & \mathbf{P}^k_{12} \\ \mathbf{P}^k_{21} g_k & \mathbf{P}^k_{22} \end{bmatrix}^{-1}.$$

Remark 3.1 *There exists a permutation matrix* $\mathbf{A} = \mathbf{A}^{-1}$ *such that* $\mathbf{P} = \mathbf{A}\mathbf{P}^k\mathbf{A}$, $\mathbf{G} = \mathbf{A}\mathbf{G}^k\mathbf{A}$, *and* $\mathbf{S}_A = [\mathbf{I} + \mathbf{P}^k\mathbf{G}^k]^{-1} = \mathbf{A}\mathbf{S}\mathbf{A}$. *Hence the* 11 *element of* $\mathbf{S}_A$ *is the* kk *element of* $\mathbf{S}$, s_{kk}.

Lemma 3.1 *Suppose that for a given* z, *the matrix* $\mathbf{M}^k(z)$ *exists, then*

$$\lim_{|g_j(z)| \to \infty,\ \forall\, j \neq k} s_{kk}(z) = \frac{1}{1 + g_k(z)\frac{\Delta(z)}{\Delta_{kk}(z)}}, \text{ and} \tag{2}$$

$$\lim_{s_{ij}(z) \to 0,\ i \neq k,\ j = 1,\ldots,m} s_{kk}(z) = \frac{1}{1 + g_k(z)\frac{\Delta(z)}{\Delta_{kk}(z)}}. \tag{3}$$

Proof: see [29], lemma 5.

Lemma 3.1 states that (2) holds for a given complex number z. However, even if (2) holds for an interval $z \in [0, jR]$, it does not follow that s_{kk}, seen as a SISO sensitivity function is restricted by the RHP poles and zeros of $g_k\Delta/\Delta_{kk}$ in (2), with the latter seen as a SISO open loop transfer function, as given in Lemma 2.2. However this restriction holds under certain conditions as will be shown now. We shall then have the result that the sensitivity reduction of s_{kk} will be restricted by the RHP poles and zeros of Δ/Δ_{kk} when g_i, $i \neq k$ are large enough.

When $\mathbf{G}$ is diagonal, the sensitivity element s_{kk} can always be expressed as

$$s_{kk} = \frac{1}{1 + g_k \sigma_k} \tag{4}$$

where σ_k is the transmission from the k:th plant input to the k:th plant output with all loops closed except the k:th. Note that σ_k does not depend on g_k. If $g_k\Delta/\Delta_{kk}$ belongs to $\mathcal{L}_1$, it is reasonable, by (3), to assume that $g_k\sigma_k$ also belongs to $\mathcal{L}_1$ when g_i are large enough for $i \neq k$. Moreover it is reasonable to identify the cross-over frequency limitations of s_{kk} with those whose open loop is $g_k\Delta/\Delta_{kk}$. The conditions for that to happen are given in the next two lemmas.

Lemma 3.2 *Let Z denote a domain that includes all the RHP poles and zeros of Δ/Δ_{kk} and an interval $[0, jR]$ on the imaginary axis for some $R > 0$. Suppose that $\mathbf{M}^k$ has no poles in Z, that g_i for $i \neq k$ do not have zeros in Z, that $1 + g_k\Delta/\Delta_{kk}$ does not have zeros in Z and that $g_k \neq 0$. Then there exist positive functions $h_i(z)$, $z \in Z$, $i \neq k$ such that*

$$\lim_{|g_i(j\omega)|\to\infty,\,\forall\, i\neq k,\, \omega\in[0,R]} \sigma_k(z) = \frac{\Delta(z)}{\Delta_{kk}(z)}, \tag{5}$$

for z belonging to any closed subdomain of Z, provided that $|g_i(z)| > h_i$ for all $i \neq k$.

Proof: see [29], lemma 6.

Note that the condition of Lemma 3.2, that $\mathbf{M}^k$ should not have poles in Z, is needed to have local uniform boundeness of s_{kk} as a function of all g_i, $i \neq k$. Considering the 11 element of equation (21) in [29] when $\mathbf{M}$ is replaced by $\mathbf{M}^k$ and $\mathbf{G}_2$ by $\mathbf{G}_2^k$, we have that

$$s_{kk} = \mathbf{M}_{11}^k - \mathbf{M}_{12}^k \left(\mathbf{G}_2^k + \mathbf{M}_{22}^k\right)^{-1} \mathbf{M}_{21}^k,$$

Therefore, the condition can be replaced by the condition that s_{kk} has no poles in Z and is locally uniformly bounded for all g_i, $i \neq k$, having large enough amplitudes. Moreover, it is shown now that the poles and zeros of σ_k approach those of Δ/Δ_{kk} for sufficiently high gain control.

Lemma 3.3 *Assume the same notation and assumptions as in Lemma 3.2. Let z_0 be a zero (pole) of Δ/Δ_{kk}. Then for any $\epsilon > 0$ there exists an $N > 0$ and $h_i(z) > 0$, $i \neq k$ which do not depend on ϵ, such that σ_k has a zero (pole) $z_\sigma \in Z$ satisfying $|z_\sigma - z_0| < \epsilon$ provided that (i) $|g_i(j\omega)| > N$ for $\omega \in [0, R]$ and $i \neq k$ and (ii) $|g_i(z)| > h_i$ for all $z \in Z$ and $i \neq k$. Moreover σ_k has the same number of poles and zeros in Z as Δ/Δ_{kk}.*

Proof: see [29], lemma 7.

The motivation for Lemma 3.3 and Corollary 2.1 is that we can now expect that if $g_i(j\omega)$, $i \neq k$ have large enough amplitudes and no RHP zeros (or equivalently s_{ij} are small enough for $i \neq k$, $j = 1, ..., m$), then the cross-over frequency limitations of s_{kk} will be as close as desired to that given in Lemma 2.2 where the RHP poles and zeros of the plant are those of Δ/Δ_{kk}. Detailed conditions for this observation to occur is phrased in Lemma 3.4, which contains a key result required for the understanding of the main Theorem 3.1.

Lemma 3.4 *Let us denote by $[\omega_1, \omega_2]$ the cross over frequency range of $l(s) = g_k\Delta/\Delta_{kk}$, where the phase margin of $l(s)$ is at least Φ as developed in Lemma 2.2. Then for any $\epsilon > 0$ there exist R, $N > 0$ and $h_i(z)$, $i \neq k$, such that the cross-over frequency of $g_k\sigma_k$ (see the definition of σ_k in (4)), is in the range $[\omega_1 - \epsilon, \omega_2 + \epsilon]$ provided that (i) $|g_i(j\omega)| > N$ for all $\omega \in [0, R]$ and $i \neq k$; (ii) $|g_i(z)| > h_i(z)$ on Z_R, a RHP domain which includes all the RHP poles and zeros of Δ/Δ_{kk} and $[0, jR]$; (iii) g_k is such that the elements of $\mathbf{M}^k$ do not have poles in Z_R; and (iv) $g_k\sigma_k$ belongs to $\mathcal{L}_1$ with phase margin of at least Φ.*

Conditions (i) and (ii) are needed in order to guarantee that the RHP poles and zeros of Δ/Δ_{kk} are close to those of σ_k. Condition (iii) is there to avoid introducing RHP zeros into σ_k which do not converge to those of Δ/Δ_{kk}, and condition (iv) is there to guarantee the applicability of Lemma 2.2. Note that a similar result under stronger conditions but without proof appears in [23]. Similar results for stable plants are given in [17, 18] with proofs in the thesis [16].

Proof: We start with $k = 1$. Let $h_i(z) > 0$, $i \neq k$, be such that s_{kk} is locally uniformly bounded in Z_R if $|g_i(z)| > h_i$ in Z_R. As in Lemma 3.2 it can be shown that such h_i's exist. Now if the lemma is not true, then there exist ϵ_0 and a sequence of matrix transfer function $\mathbf{G}_v = \mathrm{diag}(g_{i_v})$, such that (i) $|g_{i_v}(j\omega)| > v$ for all $\omega \in [0, R]$ and $i \neq k$; (ii) $|g_{i_v}(z)| > h_i$, $i \neq k$, in Z_R; and (iii) $g_{k_v}\sigma_{k_v}$ belongs to $\mathcal{L}_1$ with phase margin of at least Φ with the cross-over frequency outside $[\omega_1 - \epsilon_0, \omega_2 + \epsilon_0]$. By Lemma 3.2, σ_{k_v} converges to Δ/Δ_{kk} in Z_R. Hence by Lemma 3.3 for large enough v the poles and zeros of σ_{k_v} in Z_R are also in ϵ-neighborhood of those of Δ/Δ_{kk}. Therefore by Corollary 2.1 the cross-over frequency range of $g_k\sigma_k$ converges to a range within $[\omega_1, \omega_2]$, which is a contradiction. The extension to any k is the same as in Lemma 3.1. □

If in Lemma 3.4, Z_R includes the RHP semi-circle of radius R and R is taken large enough and the number of RHP zeros of σ_k outside Z_R is bounded, then Lemma 3.4 means that a controller, $\mathbf{G}$, which makes the sensitivity $|s_{ij}|$ very low for all $i \neq k$ and $j = 1, ..., m$, will cause $|s_{kk}|$ to be large. Moreover the sensitivity reduction of $|s_{kk}|$ will be limited close to as if it were the sensitivity of a SISO closed-loop transfer function whose open-loop is $g_k\Delta/\Delta_{kk}$; provided that the amplitudes of $g_i(z)$, $i \neq k$, are large enough for all $z \in Z_R$.

Under the conditions of Lemma 3.1 it can be proven, by the definition of $\mathbf{Q}^k$ that the sensitivity s_{kj} for $j = 1, ..., m$ is linked to s_{kk} by

$$\lim_{|g_j| \to \infty, j \neq k} \frac{s_{ku}(z)}{s_{kk}(z)} = \frac{\Delta_{ku}}{\Delta_{kk}}. \quad (6)$$

Lemma 3.4 gives upper and/or lower bounds on the cross-over frequency of $g_k\sigma_k$ as a function of its phase margin, Φ, under the condition that the sensitivity of all other elements in all rows $i \neq k$ can be made as small as required. But this condition may not be satisfied. The following is a discussion of two such cases showing that more than one row of $\mathbf{S}$ may have high sensitivity.

When Δ_{kk} has RHP zeros close to the RHP zeros of Δ it may not be possible to design a robust g_k to stabilize $1/(1 + g_k\Delta/\Delta_{kk})$ with some desired phase margin. If this is the case then one is forced to make at least one of the $|g_i|$'s for $i \neq k$ small, therefore the sensitivity of that row, in addition to the k:th row, is not low. The conclusion is to choose, if possible, the transfer matrix $\mathbf{W}$ so that Δ_{kk} has no RHP zeros or its RHP zeros are far enough from the RHP zeros of Δ.

The second case requires a preliminary lemma:

Lemma 3.5 *Let* $\mathbf{P} = \mathbf{D}^{-1}\mathbf{N}$ *where* $\mathbf{D}$ *and* $\mathbf{N}$ *are matrices of polynomials, then the denominator of any minor of* $\mathbf{P}$ *is* $\det(\mathbf{D})$ *or a polynomial that divides it (without remainder).*

Proof: Using [9], page 21, the denominator of each minor of $\mathbf{D}^{-1}$ is $\det(\mathbf{D})$ or a polynomial that divides it. Using the Binet-Cauchy formula [20] the same is true for any minor of $\mathbf{P}$. □

Note that $\Delta = \det(\mathbf{N})/\det(\mathbf{D})$. Hence, if no RHP cancellation takes place between $\det(\mathbf{N})$ and $\det(\mathbf{D})$, all the RHP poles of Δ_{kk} including multiplicity, are RHP poles of Δ. If, moreover, g_k does not contain RHP zeros, then the only possible RHP zeros of $g_k\Delta/\Delta_{kk}$ are among the RHP zeros of $\det(\mathbf{N})$.

On the other hand, if a RHP cancellation takes place between $\det(\mathbf{N})$ and $\det(\mathbf{D})$, then Δ_{kk} may have RHP poles which are not cancelled by RHP poles of Δ. These will then become RHP zeros of Δ/Δ_{kk} which may cause the same effect as described above.

3.2 Limitations for several rows of S

The previous section showed the sensitivity reduction limitations on the elements of one row of the sensitivity matrix $\mathbf{S}$, under the condition that the sensitivities of all other rows tend to zero. In this section we refine this result and identify from which rows at least one is sensitivity reduction limited. We will also show how many rows must suffer from the sensitivity reduction limitation. Let

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{11}\mathbf{G}_1 & \mathbf{P}_{12}\mathbf{G}_2 \\ \mathbf{P}_{21}\mathbf{G}_1 & \mathbf{P}_{22}\mathbf{G}_2 \end{bmatrix}, \tag{7}$$

be a partition of $\mathbf{L}$ where $\mathbf{L}_{11}$ is a $k \times k$ transfer matrix and $\mathbf{G} = \mathrm{diag}(\mathbf{G}_1, \mathbf{G}_2)$. Consider the feedback system described schematically in Fig. 2 whose transfer matrix from $\mathbf{r}_1$ to $\mathbf{y}_1$ is

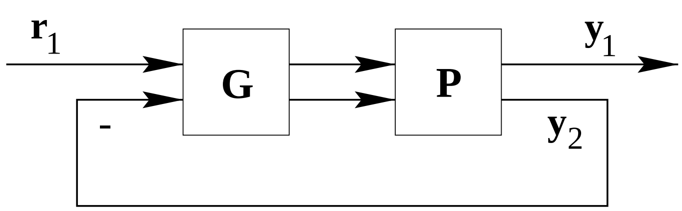

Figure 2: The partitioned MIMO feedback structure

$$\begin{aligned} \tilde{\mathbf{L}}_{11} &= \mathbf{L}_{11} - \mathbf{L}_{12}[\mathbf{I} + \mathbf{L}_{22}]^{-1}\mathbf{L}_{21} \\ &= \mathbf{L}_1 \begin{bmatrix} \mathbf{I} \\ -[\mathbf{I} + \mathbf{L}_{22}]^{-1}\mathbf{L}_{21} \end{bmatrix}, \end{aligned} \tag{8}$$

where

$$\mathbf{L}_1 = [\mathbf{L}_{11}\ \mathbf{L}_{12}] = [\mathbf{P}_{11}\mathbf{G}_1\ \mathbf{P}_{12}\mathbf{G}_2]. \tag{9}$$

Lemma 3.6 *Suppose (i) that* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *are full rank and such that* $\tilde{\mathbf{L}}_{11}$ *is full rank, and (ii) that a RHP zero of* $[\mathbf{P}_{11}\ \mathbf{P}_{12}]$ *is not a pole of* $[\mathbf{P}_{11}\ \mathbf{P}_{12}]$ *or a pole of* $\mathbf{L}_1$ *or a pole of* $\tilde{\mathbf{L}}_{11}$. *Then each RHP zero of* $[\mathbf{P}_{11}\ \mathbf{P}_{12}]$ *is a RHP zero of* $\tilde{\mathbf{L}}_{11}$.

Proof: By assumption (i) each RHP zero of $[\mathbf{P}_{11}\ \mathbf{P}_{12}]$ is a common RHP zero of each $k \times k$ minor of $\mathbf{L}_1$. Hence by the Binet-Cauchy formula [20], and the assumption (iii) in Section 1.1 it is a zero of $\det(\tilde{\mathbf{L}}_{11})$, thus a pole of its inverse. Now use [5]. □
Note that the conditions of Lemma 3.6 are generically satisfied.

Let

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \tag{10}$$

be a partition of the sensitivity function $\mathbf{S}$ for the closed loop system of Fig. 1 where $\mathbf{S}_{11}$ is a $k \times k$ matrix. Then it is easily shown, using the block matrix inversion formula of [20], page 656, that

$$\mathbf{S}_{11} = (\mathbf{I} + \tilde{\mathbf{L}}_{11})^{-1} \tag{11}$$

i.e. the upper left $k \times k$ block of the sensitivity function of the closed loop system of Fig. 1 equals the sensitivity function of a closed loop whose open loop is the transfer matrix from $\mathbf{r}_1$ to $\mathbf{y}_1$ in Fig. 2.

By Lemma 3.6 the transfer matrix of Fig. 2 from $\mathbf{r}_1$ to $\mathbf{y}_1$ includes the RHP zeros of $[\mathbf{\Pi}_{11}\ \mathbf{\Pi}_{12}]$ which cannot be made to disappear in $[\mathbf{P}_{11}\ \mathbf{P}_{12}] = [\mathbf{\Pi}_{11}\ \mathbf{\Pi}_{12}]\mathbf{W}$ by any choice of proper $\mathbf{W}$ due to the assumption (ii) in Section 1.1. Thus $\tilde{\mathbf{L}}_{11}$ obeys the sensitivity reduction limitations as developed above.

This means that if rows $1, ..., r$ of the plant $\mathbf{\Pi}$ form an NMP transfer matrix $\mathbf{\Pi}_1$, then at least one of the sensitivity functions s_{ii} $i = 1, \ldots, r$ of the system of Fig. 1 must suffer from gain reduction limitations due to RHP poles and zeros included in $\mathbf{\Pi}_1$. The same is true of course for any transfer matrix, $\mathbf{\Pi}_r$, formed from the r rows of $\mathbf{\Pi}$ specified by the index set $i_1, ..., i_r$. This forms the basis of the proof of the following theorem which is our main result, where the following notation is used: $\mathbf{P}^r$ and $\mathbf{G}^r$ are the matrix transfer functions obtained when rows and columns $i_1, ..., i_r$ of $\mathbf{P}$ and $\mathbf{G}$ change places with the first r rows and columns, respectively. Moreover, $\tilde{\mathbf{L}}_{11}^r$ is formed from the plant $\mathbf{P}^r$ and controller $\mathbf{G}^r$ exactly as $\tilde{\mathbf{L}}_{11}$ is formed from $\mathbf{P}$ and $\mathbf{G}$.

Theorem 3.1 *Consider the system in Fig. 1. Suppose that (i) it is closed-loop stable; (ii)*

$$\mathbf{\Pi}_r = \left[\pi_{i_1}^T \ldots \pi_{i_r}^T\right]^T, \quad \pi_q = \text{row } q \text{ of } \mathbf{\Pi}$$

is a NMP transfer matrix formed from rows $i_1, ..., i_r$ of $\mathbf{\Pi}$; and (iii) $\tilde{\mathbf{L}}_{11}^r$, $\mathbf{P}^r$ and $\mathbf{G}^r$ satisfy the conditions of Lemma 3.6. Then at least one of loop transmission $g_k\sigma_k$, $k = i_1, ..., i_r$, see (4), must suffer from cross-over frequency limitations as in Lemma 3.4 applied on the transfer matrix $\mathbf{\Pi}_r$.

An immediate consequence is that the greater the drop of the rank is, by the RHP zeros of the plant, the more restrictive the contraints become. For example if in a 4×4 plant, rows $\{1, 2\}$ and $\{3, 4\}$ drop rank by the same or different zeros, then at least two of the sensitivity functions suffer from gain reduction limitations. This consequence includes that of [10], theorem 2.2 ff which is related to a particular zero and where the condition is stated as an integral inequality.

Theorem 3.1, leads to the following guidelines, some of which are commonly known, when designing NMP MIMO feedback systems.

- For each NMP block $\mathbf{\Pi}_r$, at least one of the rows $i_1, ..., i_r$ of the sensitivity matrix, $\mathbf{S}$, will be subject to sensitivity reduction limitations.

- It is possible, in general, to select the row(s) of $\mathbf{S}$ that will suffer from high sensitivity, and by sacrificing its sensitivity, to achieve the desired sensitivity from the other rows. For example: If in a 3×3 plant, rows $1, 2$ form a NMP transfer matrix and also rows $1, 3$; then row 1 can be sacrificed to have high sensitivity while the other rows can have low sensitivity.

- If the sensitivity of a maximum possible number of rows of $\mathbf{S}$ is high then the sensitivity of the other rows must be low. The sensitivity reduction is quantified by a SISO transfer function whose cross-over frequency is in a range which is bounded from above and below by a simple function of the RHP poles and zeros of the plant.

- Increasing the number of inputs to a plant, may change a NMP plant to a minimum-phase one, and thus overcome the closed-loop sensitivity reduction limitations due to RHP zeros.

4 An example

Consider the plant

$$\mathbf{\Pi} = \begin{bmatrix} \frac{0.25}{s+1} & \frac{2}{s+5} & \frac{0.5}{s+1} \\ \frac{0.5}{s+1} & \frac{2}{s+1} & \frac{1}{s+1} \\ \frac{0.875}{s+6} & \frac{2/3}{s+1} & \frac{0.5}{s+1} \end{bmatrix}.$$

The first two rows have one zero at +3, the first and third rows have one zero at +1, the last two rows have no zero, while the full plant has zeros at +3, +1, and -1. Clearly, no single row has any zero.

Referring to figure 1, let the feedback compensator be $\mathbf{WG}$, where we chose

$$\mathbf{W} = \begin{bmatrix} 4 & 0 & 0 \\ \frac{7.5(s-1)}{s+6} & 3 & -1 \\ \frac{-17(s-0.1765)}{s+6} & -4 & 2 \end{bmatrix}$$

to be stable and NMP. Then

$$\mathbf{P} = \mathbf{\Pi W} = \begin{bmatrix} \frac{7.5(s-3)(s-1)}{(s+5)(s+6)(s+1)} & \frac{4(s-1)}{(s+5)(s+1)} & \frac{-(s-3)}{(s+5)(s+1)} \\ 0 & \frac{2}{s+1} & 0 \\ 0 & 0 & \frac{1/3}{s+1} \end{bmatrix}.$$

While no row, or combination of two rows of $\mathbf{P}$ has any RHP zero, $\mathbf{P}$ itself retains the zeros at $+1$, and $+3$, located in the elements of its first row. By letting $\mathbf{G}$ be diagonal, it is possible to achieve arbitrary cross over frequency, and, for a given frequency, arbitrary small closed loop sensitivity for rows two and three, and restricting the cross-over frequency and sensitivity reduction limiting effects of the RHP zeros to the first row, only. A simple choice is $\mathbf{G} = \text{diag}(0.29, 10, 60)$.

5 Conclusions

In this paper, limitations on the achievable sensitivity reduction via feedback is given, for non-minimum phase MIMO plants for which the number of inputs is at least as large as the number of outputs. It is shown that if r rows of the plant forms an NMP transfer matrix then the achievable reduction of the sensitivity gains of one or more of the corresponding rows of the full closed loop transfer matrix is limited. The sensitivity reduction constraint is quantified with the help of the achievable cross-over frequency of a specially constructed SISO transfer function that includes the right-half-plane zeros and poles of the r-row NMP transfer matrix. The results in this paper may serve as a valuable design tool for the sensitivity trade-off in non-minimum phase MIMO systems, based on the cross over frequency criterion of the loop transmission in different channels.

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