On smooth optimal control determination

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Abstract. When using the Pontryagin Maximum Principle in optimal control problems the most difficult part of the numerical solution is associated with the nonlinear operation of the maximization of the Hamiltonian over the control variables. For a class of problems, the optimal control vector is a vector function with continuous time derivatives. A method is presented to find this smooth control without the maximization of Hamiltonian. Three illustrative examples are considered.

1. Description of the method

Let us consider the classical optimal control problem (OCP), Pontryagin *et. al.* 1962, Lee and Markus, 1967, Athans and Falb, 1966, Pinch, 1993, in the form

$$\int_0^T f_0(x,u)dt \to \min \tag{1}$$

$$\frac{dx}{dt} = f(x,u), \tag{2}$$

$$x(0) = x_0,\ \ x(T) = x_T. \tag{3}$$

where the control variables $u(t) \in \mathbf{R^m}$, the state variables $x(t) \in \mathbf{R^n}$, and $f(x,u) \in \mathbf{R^n}$ are column vectors, with $m \leq n$. The right-hand side functions of the state equations (2), $f(x,u)$ and the performance index function $f_0(x,u)$ are smooth over all arguments. It is assumed that that there are no other constraints except of (3). The column vector $p(t) \in \mathbf{R^n}$ is the vector of costate variables. According to Pontryagin's Maximum Principle (PMP), Pontryagin *et al.* (1962), the Hamiltonian H, is defined as

$$H = p^T f(x,u) - p_0 f_0(x,u). \tag{4}$$

Let us consider the "normal" case when $p_0 > 0$. Without loss of generality we set $p_0 = 1$, leading to

$$H = p^T f(x,u) - f_0(x,u). \tag{5}$$

According to PMP the following system of differential equations for the co-state variables p holds

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}^T = -\frac{\partial f}{\partial x}^T p + \frac{\partial f_0}{\partial x}^T \tag{6}$$

If there exists an optimal solution (x*, u*), then, according to PMP, there exists a costate vector p^* such that the following conditions are satisfied:

(a) $H(x*, u*, p*) \geq H(x*, u, p*)$, meaning that H has maximum over the control u,

(b) the variables x*, p* satisfy equations (2), (6),

(c) the end conditions (3) must be hold.

In order to find the solution (x*,u*,p*) from PMP, the two-point boundary problem for the system of equations (2)-(6) must be solved with the end conditions (3). At each point in time, u is determined from the condition (a). There are as many unknown values $p(0)$ as given end conditions $x(T) = x_T$. The maximization problem in (a) is often computationally costly.

Since no other constraint than (3) is present, it follows from (a) that the control vector u^* has to satisfy the necessary conditions

$$\frac{\partial H}{\partial u} = 0. \tag{7}$$

Often these equations are nonlinear in u^*. In the following, we shall omit the symbol * for simplicity. From (5) and (7) one gets the system of equations

$$p^T \frac{\partial f}{\partial u} - \frac{\partial f_0}{\partial u} = 0, \tag{8}$$

where $\partial f_0/\partial u$ is a row vector, and the dimension of the Jacobian $\partial f/\partial u$ is $n \times m$. The equations (8) are linear in p. As mentioned above, according to the usual procedure for solving an OCP with PMP, the nonlinear system of m equations (8) must be solved in order to find the control vector u as a function of the costate vector p and the state vector x. In contrast, by linear operations on (8) it is very easy to compute m elements of the costate vector p as a vector function of the state vector x, the control vector u, and the $n - m$ remaining costates. Let us assume that the rank of the Jacobian matrix $\partial f/\partial u$ is equal to m. This means that there exists a non-singular $m \times m$ sub-matrix. We shall re-index the state variables in such a manner that the m variables corresponding to this sub-matrix have indices 1 to m, and we shall denote the sub-vectors of the corresponding right-hand functions in (2) as f^a, the costates as p^a, the sub-vector of the remaining right-hand functions as f^b, and the remaining costates as p^b. Consequently,

$$x = [x^a; x^b]; p = [p^a; p^b]; \; f(x,u) = [f^a(x,u); f^b(x,u)].$$

The equations (8) can be rewritten as

$$p^{aT} \frac{\partial f^a}{\partial u} + p^{bT} \frac{\partial f^b}{\partial u} - \frac{\partial f_0}{\partial u} = 0, \tag{9}$$

From here follows that

$$p^a = -(\frac{\partial f^a}{\partial u}^T)^{-1} \frac{\partial f^b}{\partial u}^T p^b + (\frac{\partial f^a}{\partial u}^T)^{-1} \frac{\partial f_0}{\partial u}^T \doteq A(x, p^b, u), \tag{10}$$

where for convenience we have defined the vector A. Taking the derivative of the co-state p^a from (10) with respect to time t, one gets

$$\frac{dp^a}{dt} = \frac{\partial A}{\partial x} f(x,u) + \frac{\partial A}{\partial u}\frac{du}{dt} + \frac{\partial A}{\partial p^b}\frac{dp^b}{dt} \tag{11}$$

The differential equations for the costates (6) can be rewritten as

$$\frac{dp^a}{dt} = -\frac{\partial H}{\partial x^a}^T = -\frac{\partial f^a}{\partial x^a}^T p^a - \frac{\partial f^b}{\partial x^a}^T p^b + \frac{\partial f_0}{\partial x^a}^T, \tag{12}$$

$$\frac{dp^b}{dt} = -\frac{\partial H}{\partial x^b}^T = -\frac{\partial f^a}{\partial x^b}^T p^a - \frac{\partial f^b}{\partial x^b}^T p^b + \frac{\partial f_0}{\partial x^b}^T. \tag{13}$$

By substituting the expressions for p^a from (10) into the right-hand sides of (12, 13), and exploiting the fact that the right-hand side of (12) is equal to the right-hand side of (11), one gets, with a simple matrix transformation, the equations for the time derivatives du/dt as functions of x, p^b, and u only, as follows. Denoting

$$B \doteq \frac{\partial A}{\partial u}$$

we shall assume that B is non-singular. Thus,

$$\frac{\partial A}{\partial x} f(x,u) + \frac{\partial A}{\partial p^b}\frac{dp^b}{dt} + B\frac{du}{dt} = -\frac{\partial f^a}{\partial x^a}^T p^a - \frac{\partial f^b}{\partial x^a}^T p^b + \frac{\partial f_0}{\partial x^a}^T,$$

and hence

$$\frac{du}{dt} = B^{-1}[-\frac{\partial f^a}{\partial x^a}^T A - \frac{\partial f^b}{\partial x^a}^T p^b + \frac{\partial f_0}{\partial x^a}^T - \frac{\partial A}{\partial x} f(x,u) - \frac{\partial A}{\partial p^b}\frac{dp^b}{dt}] \doteq F(x,p^b,u), \tag{14}$$

where the notation $F(x,p^b,u)$ is defined. Substituting the expression for $A(x,p^b,u)$ from (10) into the right-hand side of (13), we shall denote the resulting expression as $S(x,p^b,u)$. Thus we have

$$\frac{dp^b}{dt} = S(x,p^b,u).$$

Thus, the problem is reduced to integrating the system of equations

$$\begin{aligned} \frac{du}{dt} &= F(x,p^b,u), \\ \frac{dx}{dt} &= f(x,u), \\ \frac{dp^b}{dt} &= S(x,p^b,u). \end{aligned} \tag{15}$$

with the initial values $x(0) = x_0$, and the appropriate initial values $u(0) = u_0, p^b(0) = {p^b}_0$ to be found. The end conditions $x(T) = x_T$ have to be satisfied. The number of integrated equations (15) is the same as in the PMP algorithm, namely $2n$, where instead of the differential equations for p^a we have the differential equations for u. Note that no maximization of the Hamiltonian, nor any non-linear operations are needed for the determination of $u(t)$. Instead, just a number of linear matrix operations have to be performed, which is a much easier task.The following theorem can be formulated based on the notations and derivations given above.

Theorem. If the OCP problem (1-3), $m \le n$, has the optimal solution x^*, u^* such that u^* is smooth and belong to the open set U, and if also the problem is normal in the sense that the costate p_0^* can be set to 1, the Jacobian $\partial f^a/\partial u$ is non-singular, the Jacobian B is non-singular, then the optimal states, x^*, costates, ${p^*}^b$, and control u^* satisfy the equations (15).

Note. In the case when $n = m$ the set of variables and costates x^b, p^b is empty and the equations (15) are changed accordingly. In this case the last set of these equations, namely those related to p^b, vanishes.

2. Rigid body rotation

As the first example, let us consider the following axisymmetric rigid body rotation problem (Athans and Falb, 1963):

$$\begin{aligned} \frac{dx}{dt} &= ay + u_1, \\ \frac{dy}{dt} &= -ax + u_2 \end{aligned} \tag{16}$$

The variables x and y are the components of the angular velocity that need to be stopped, *i.e.* the final conditions are

$$x(T) = 0, y(T) = 0, \tag{17}$$

and final time T is given. The performance index to be minimized is chosen as

$$J = \frac{1}{4}\int_0^T (u_1^2 + u_2^2)^2 dt \rightarrow \min$$

Here, according to (10), the vector A has the form

$$p = A = \left[\begin{array}{c} u_1(u_1^2 + u_2^2) \\ u_2(u_1^2 + u_2^2) \end{array} \right] \tag{18}$$

The costate equations are

$$\begin{aligned} \frac{dp_x}{dt} &= ap_y \\ \frac{dp_y}{dt} &= -ap_x \end{aligned} \tag{19}$$

One may notice that the Hamiltonian is strictly concave. Here $\partial A/\partial x = 0$. The matrix B is symmetric and has the form

$$B = \left(\begin{array}{cc} 3u_1^2 + u_2^2 & 2u_1u_2 \\ 2u_1u_2 & u_1^2 + 3u_2^2 \end{array} \right) \tag{20}$$

with

$$B^{-1} = \frac{1}{(3[u_1^2 + u_2^2]^2)} \left(\begin{array}{cc} u_1^2 + 3u_2^2 & -2u_1u_2 \\ -2u_1u_2 & 3u_1^2 + u2^2 \end{array} \right) \tag{21}$$

Extracting the time derivatives for u_1, u_2 one gets after some simplifications

$$F = \left[\begin{array}{c} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{array} \right] = \left[\begin{array}{c} au_2 \\ -au_1 \end{array} \right] \tag{22}$$

implying

$$\begin{aligned} \frac{du_1}{dt} &= au_2, \\ \frac{du_2}{dt} &= -au_1 \end{aligned} \tag{23}$$

From (23) it is easily seen that

$$u_1^2 + u_2^2 = C^2 = const$$

Hence the control vector $(u_1, u_2)^T$ has constant length $C > 0$, and it rotates around the origin of the coordinate system in the u-plane with the constant angular velocity a. Introducing the variables r, θ by the coordinate transformation

$$\begin{aligned} x &= r\sin\theta \\ y &= r\cos\theta \end{aligned} \tag{24}$$

one realizes that the system of equations (16), (23) has the solution

$$\begin{aligned} u_1 &= -C\frac{x}{\sqrt{x^2 + y^2}} \\ u_2 &= -C\frac{y}{\sqrt{x^2 + y^2}} \end{aligned} \tag{25}$$

Thus, the initial value of the vector $u = [u_1; u_2]$ is collinear and has opposite direction to the initial vector of the state coordinates [x;y]. To check the admissibility of (25), one may substitute (24) into (16) and find the equations for the time derivatives of r, θ. To find the derivative of r, the resulting equations must be multiplied by $\sin\theta, \cos\theta$, respectively, and then summed. To find the derivative of θ the same equations must be multiplied by $\cos\theta$ and $-\sin\theta$, respectively, and then summed. Substituting (25) into the obtained equations one gets

$$\begin{aligned} \frac{dr}{dt} &= -C \\ \frac{d\theta}{dt} &= a. \end{aligned} \tag{26}$$

Thus one can see that both vectors $(u_1, u_2)^T$ and $(x, y)^T$ rotate with the same angular velocity a and their collinearity is maintained in steady-state.

Let us denote $\sqrt{x(0)^2 + y(0)^2} = R$. From the final conditions (17), and the first of equations (26), one easily gets that $u_1(0)/C = -x(0)/R$, $u_2(0)/C = -y(0)/R$, and $C = R/T$. Hence, the problem is completely solved without maximization of the Hamiltonian.

3. Optimal spacing for greenhouse lettuce growth

Let us consider the problem of the optimal variable spacing policy for greenhouse lettuce growth, see Ioslovich and Gutman, 1999; Seginer and Ioslovich, 1999; Ioslovich and Gutman, 2000. The dynamical model of the lettuce growth in constant climate conditions has the form

$$\frac{dv}{dt} = aG(W), \tag{27}$$

where v is the dry mass of a single plant $[kg/m^2]$, a is the spacing $[m^2/plant]$, G is the net photosynthesis $[kg/m^2]$, W is the plant density $[kg/m^2]$. The net photosynthesis $G(W)$ is a strictly concave function. The spacing is variable in time and must be adjusted in an optimal way during the growth process to minimize the cost of the process, which is set as

$$J = \int_0^T a c_R dt \tag{28}$$

where $c_R[\$/m^2 s]$ is the price of rent, including operational costs. The spacing and plant density are connected via

$$v = aW, \tag{29}$$

thus plant density W can be considered as a control variable instead of the spacing a. The state equation and the expression for the performance index can be rewritten in the form

$$\begin{aligned} \frac{dv}{dt} &= \frac{v}{W} G(W), \\ J &= \int_0^T \frac{v}{W} c_R dt \end{aligned} \tag{30}$$

The final time T is free and the final moment T is determined through the condition

$$v(T) = v_T, \tag{31}$$

where v_T is a given final marketable weight of a single lettuce plant. The Hamiltonian, H, has the form

$$H = \frac{v}{W}(pG(W) - c_R), \tag{32}$$

where p is the scalar co-state variable. The maximization of the Hamiltonian over the control W gives

$$\frac{\partial H}{\partial W} = -\frac{v}{W^2}(pG(W) - c_R) + \frac{vp}{W}\frac{\partial G}{\partial W} = 0 \tag{33}$$

From here the value of p is found as

$$p = \frac{c_R}{G(W) - W\frac{\partial G}{\partial W}} \tag{34}$$

The differential equation for the co-state p has form

$$\frac{dp}{dt} = -\frac{\partial H}{\partial v} = -\frac{pG(W) - c_R}{W} \tag{35}$$

Substituting equation for p from (34) to (35) one gets after some transformations

$$\frac{dp}{dt} = -\frac{c_R\frac{\partial G}{\partial W}}{(G - W\frac{\partial G}{\partial W})} \tag{36}$$

Differentiating (34) with respect to t one gets

$$\frac{dp}{dt} = \frac{dW}{dt}\frac{c_R W\frac{\partial G^2}{\partial W^2}}{(G - W\frac{\partial G}{\partial W})^2} \tag{37}$$

From (36) and (37) one gets the differential equation for the control variable W as follows:

$$\frac{dW}{dt} = -\frac{\frac{\partial G}{\partial W}(G - W\frac{\partial G}{\partial W})}{W\frac{\partial G^2}{\partial W^2}} \tag{38}$$

For the free final time we have the transversality condition

$$H(T) = 0 \tag{39}$$

Substituting the expression for p (34) and the expression for H (32) into the equation (39) one gets for the final moment T the following equality

$$v\frac{c_R\frac{\partial G}{\partial W}}{(G - W\frac{\partial G}{\partial W})} = 0 \tag{40}$$

From (40) and (38) it follows that there exists an optimal steady state solution with the constant control value W^* which satisfies the condition

$$\frac{\partial G(W)}{\partial W} = 0 \tag{41}$$

Thus, the optimal lettuce density is constant and corresponds to the maximal net photosynthesis. The problem is solved without maximization of the Hamiltonian, which in this case seems rather simple, but unexpected in advance. However, if the final time is not free, and is fixed *a priori* at some non-optimal value t_f, the equation (38) makes is possible to avoid the maximization of the Hamiltonian, which constitutes a difficult numerical problem for each time step. For details, see Ioslovich and Gutman, 1999.

4. Maximal area surrounded by a curve of given length

This is very well known problem described in *e.g.* Gelfand and Fomin, 1969, as a variational, isoperimetric problem. We shall solve it in the case of given parameters as an OCP.

$$\int_0^1 x_1 dt \to \max \tag{42}$$

$$\frac{dx_1}{dt} = u, \tag{43}$$

$$\frac{dx_2}{dt} = \sqrt{1 + u^2}. \tag{44}$$

The considered end conditions are

$$x_1(0) = x_1(1) = 0,\ x_2(0) = 0,\ x_2(1) = \pi/3. \tag{45}$$

The Hamiltonian has the form

$$H = p_1 u + p_2\sqrt{1+u^2} + x_1. \tag{46}$$

The differential equations for the costate variables are

$$\begin{aligned} \frac{dp_1}{dt} &= -1, \\ \frac{dp_2}{dt} &= 0. \end{aligned} \tag{47}$$

From (46) one gets

$$\frac{\partial H}{\partial u} = p_1 + p_2\frac{u}{\sqrt{1+u^2}} = 0. \tag{48}$$

In fact (48) is possible to solve with respect to u. However this is not always the case in general, and we want to demonstrate our new method. Here we use p_1 as p_a and p_2 as p_b. From (48) the value of p_1 is found in the form

$$p_1 = -p_2\frac{u}{\sqrt{1+u^2}}. \tag{49}$$

Taking the derivative with respect to t of (49) and using (47) one gets

$$\frac{du}{dt} = \frac{(1+u^2)^{3/2}}{p_2}. \tag{50}$$

Now, taking into account that p_2 is a constant according to (47), we have to integrate numerically the equations (43), (44), (50) in order to find the unknown constants p_2 and $u(0)$ such that the conditions (45) are satisfied. The answer is $p_2 = -1$, and $u(0) = 1/\sqrt{(}3)$. The resulting curve $x_1(t)$ is a part of a circle. As in the other examples, we did not have to solve the nonlinear equation (48).

5. Conclusion

A method for extracting smooth optimal control for a class of optimal control problems was presented and three illustrative examples were shown. The method does not require the maximization of the Hamiltonian over the control, and instead substitute the ODE for the part of costates by the ODE for the smooth control.

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References.

1. Pontryagin L., V. Boltyansky, R. Gamkrelidze, E. Mischenko, 1962, Mathematical Theory of Optimal Processes, Wiley-Interscience, NY.
2. Athans M., P. Falb, 1966, Optimal Control, McGrow-Hill, NY.
3.Gelfand I.M., S.V. Fomin, 1969, Calculus of Variations, Prentice-hall, Englwood Cliffs, NJ. 4. Ioslovich I., P.-O. Gutman, 1999, Plant Factory Optimization, 3rd International Conference on Industrial Automation, Montreal, Canada, International Association for Industrial Automation (IAIA), Conference Proceedings, pp. 15.5–15.8.
5. Ioslovich I., P.-O. Gutman, 2000, Optimal Control of Crop Spacing in a Plant Factory, Automatica, 36: 1665–1668.
6. Lee E.B., L. Markus, 1967, Foundations of Optimal Control Theory, John Wiley&Sons, NY.
7. Pinch, E.R., 1993, Optimal Control and the Caluclus of variations, Oxford Science Publication, UK.
8. Seginer I., Ioslovich I., 1999, Optimal Spacing and Cultivation Intensity for an Industrialized Crop Production System, Agricultural Systems, 62: 143–157.