Robust Adaptive Output-feedback Tracking for a Class of Nonlinear Time-delayed Plants

Boris Mirkin and Per-Olof Gutman

***Abstract*— Within the model reference adaptive control (MRAC) framework, a continuous adaptive output-feedback control scheme is developed for a class of nonlinear SISO dynamic systems with time delays which is robust with respect to unknown time-varying plant delays and to an external disturbance with unknown bounds. A special form of the Lyapunov-Krasovskii functional with a "virtual" adaptation gain vector is introduced to prove stability.**

***Index Terms*— Robust adaptive control, nonlinear time-delay systems.**

I. INTRODUCTION

The adaptive control technique applied to uncertain systems with time-delays is a research area that is receiving considerable attention during the last few years, see e.g. the recent papers [1]-[17] and the references therein. Many important results have been obtained for linear [1], [3], [4], [8], [10], [12], [13], [14], [17], switched [16] and nonlinear [5], [6], [9], [13], [14] state or/and input delay plants; based on state [1], [5], [10], [11], [13], [15], [14], [17] or output [2], [3], [4], [6], [7], [8], [12] feedback; with continuous [1], [3]-[9], [17] or discontinuous [2], [10]-[14], [16] control actions.

The designs have been based mainly on the following approaches: the backstepping technique [9], [13], [15]; functional differential inclusions [2], [7]; model reference adaptive control (MRAC) [1], [3], [4], [5], [8], [17]; variable structure/sliding mode technique [10], [11], [13], [14], [16]; neural networks [11]; differential geometry tools [6] and their combinations.

In particular, adaptive state feedback stabilizing controllers were synthesized in [1] for linear plants with nonlinear delayed state perturbations bounded by a linear function of the delayed state with unknown gains whereby the delays were not assumed to be known. Also the paper [10] deals with the state feedback stabilization of linear systems with known state delays subject to a bounded external disturbance with unknown bounds. The design was based on variable structure systems theory and guarantees convergence to a small ball. The problem of output reference model signal tracking by state feedback for linear systems with a perturbation which is a bounded non-linear function of the states with multiple constant delays was considered in [5], where the reference model was, however, an autonomous dynamic linear system.

State feedback sliding mode adaptive controllers for state delayed nonlinear systems with unknown delays are considered in [11], [13], [14]. An adaptive neural control technique is used in [11] where the plant model has unknown nonlinear dead-zones and unknown gain signs. In [13] and [14] the design is based on the the backstepping and MRAC methods, respectively. Within the framework of the backstepping technique a state feedback stabilizing controller was proposed in [15] for a class of nonlinear time delay systems, where there are the linear mismatching terms but with known system matrices.

For a class of linear plants with known input delay, the design of output control in [3] was based on a Smith predictor-like structure. A state-feedback Lyapunov-based design of MRAC was considered in [17] for a class of linear systems with known input and state delays based only on lumped delays without so-called distributed-delay blocks. The design procedure is based on the concept of reference trajectory prediction, and on the formulation of an augmented error. In the framework of functional differential inclusions, [2] and [7] consider output adaptive tracking control of a class of linear, minimum-phase, SISO and MIMO systems of relative degree one, described by functional differential equations. The paper [6] used tools from differential geometry for a tracking problem for a special type of nonlinear plants.

In [12], within the model reference adaptive control framework, a simple adaptive output-feedback control scheme which uses feedback action only, and thus does not require a direct measurement of the command or disturbance signals, was developed for a class of linear SISO dynamic systems with state delays and an unknown external disturbance.

For the adaptive output feedback strategy $u(t) = \theta^T(t)\omega(t) - k_I \mathrm{sgn}(e(t))\int_0^t |e(t)|\,dt$ with the adaptation gain $\theta(t)$, the input signals $e(t) = y(t) - y_r(t)$ and $\omega(t) = [e(t)\,x_1(t)\,x_2(t)]^T$, where $x_1(t)\,x_2(t)$ are the state of the standard filters driven by the output of plant $y(t)$ and the control signal $u(t)$, e.g. [18] ensures bondedness of all signals in the closed-loop system and convergence to zero of the tracking error. This adaptive control scheme has additional benefits: disturbance rejection and robustness with respect to multiple unknown time-varying plant delays. But the price paid in this controller configuration lies primarily in the discontinuous nature of the feedback.

In the present note, an adaptive output feedback tracking problem for a special class of nonlinear plants is formulated and solved following the direct model reference path of [12]. As in [12], the new control strategy ensures the convergence to zero of the tracking error, but in contrast to [12], it is now possible to `(i)` handle a larger class of systems with uncertain delayed nonlinearities and `(ii)` avoid discontinuous control. This adds generality, together with the relative simplicity of the control strategy. To the best of our knowledge, the problem considered in this note belongs to an adaptive control area which has not previously been investigated in the framework of the direct adaptive approach.

II. PROBLEM STATEMENT

In this section we formulate the control problem, including the state delay plant model and the reference model, assumptions and the control objective. The uncertain single-input, $u(t)$, single-output, $y(t)$, nonlinear continuous-time plant with time delays appropriately initialized is of the form

$$\begin{aligned}\dot{x}(t) =& Ax(t) + A_\tau x(t-\tau_x(t)) + bu(t) + bf(y(t), y(t-\tau_y(t)), t) + bd(t)\\ x(\vartheta) =& x_0,\ \vartheta \in [-\tau_{max}, 0]\\ y(t) =& c^T x(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t), u(t) \in \mathbb{R}$, $f(y(t), y(t-\tau_y(t)), t) \in \mathbb{R}$ and $d(t) \in \mathbb{R}$ are, respectively, the state, output, control input, nonlinear delayed perturbation and external disturbance. The constant matrices A, A_τ and the vectors b and c of appropriate dimensions have unknown elements. The time-delays $\tau_x(t)$ and $\tau_y(t)$ are nonnegative differentiable functions, satisfying

$$\begin{aligned}0 \le \tau_x(t) \le \tau_{xmax},\ \dot{\tau}_x(t) \le \tau_x^* < 1\\ 0 \le \tau_y(t) \le \tau_{ymax},\ \dot{\tau}_y(t) \le \tau_y^* < 1\end{aligned} \tag{2}$$

where τ_{xmax}, τ_x^*, τ_{ymax} and τ_y^* are some unknown, positive constants. Hence, the time delays are uncertain within unknown upper bounds.

The specification includes that all signals of the closed loop system remain bounded, and that the plant output $y(t)$ asymptotically exact

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follow the output $y_r(t)$ of a stable reference model without delay

$$\dot{x}_r(t) = A_r x_r(t) + b_r r(t) \quad y_r(t) = c_r^T x_r(t), \tag{3}$$

where $x_r \in \mathbb{R}^n$, $r(t), y_r(t) \in \mathbb{R}$ and $r(t)$ is the reference input which is postulated to be a uniformly bounded and piecewise continuous function of time. The transfer function of the reference model is expressed as

$$W_r(s) = c_r^T (sI_n - A_r)^{-1} b_r = k_r \frac{N_r(s)}{D_r(s)} \tag{4}$$

where $N_r(s), D_r(s)$ are monic polynomials and k_r is a constant. Asymptotic tracking is demanded, i.e. $\lim_{t\to\infty} e(t) = 0$, where $e(t) = y(t) - y_r(t)$.

The following assumptions are made on the plant (1) and the reference model (3): **(A1)** When there are no terms with time delays τ_x, τ_y and external disturbance, the plant (1) can be described by

$$y = W_0(s)u, \quad W_0(s) = c^T (Is - A)^{-1} b = k_p \frac{N(s)}{D(s)} \tag{5}$$

where $W_0(s)$ is the transfer function associated with the undelayed, undisturbed plant; $D(s)$ is a monic polynomial of degree n; the polynomial $N(s)$ is monic and Hurwitz and of degree $n-1$, i.e. the plant is minimum phase; the high-frequency plant gain k_p is constant with known sign; **(A2)** The transfer function $W_r(s)$ is strictly positive real (SPR); **(A3)** $|d(t)| \le d^*$, where d^* is unknown and **(A4)** $A_\tau = b a_\tau^{*T}$. These assumptions are the same as in [12].

In addition to (A1)-(A4) we include the following assumption for the nonlinear function $f\big(y(t), y(t-\tau_y(t)), t\big)$. **(A5)** The function $f\big(y(t), y(t-\tau_y(t)), t\big)$ is assumed to be bounded by some polynomial, whose order p is known

$$|f(\cdot)| \le \sum_{l=1}^{p} \xi_{1l}^* |y(t)|^l + \sum_{l=1}^{p} \xi_{2l}^* |y(t-\tau_y(t))|^l \tag{6}$$

where ξ_{1l}^* and ξ_{2l}^* are *unknown* positive constants.

Remark 1: The minimum phase assumption is fundamental in MRAC schemes, see [18], [19]. The assumption that the relative degree be one focusses on the simplest case amenable to Lyapunov designs; however, the paper's main idea of the *new control parametrization* can be extended to higher relative degree, for which case it is required to use "error augmentation" and/or "tuning error normalization", see [18].

III. Adaptive Controller Parametrization

We propose the following controller structure

$$u(t) = \theta^T(t)\omega(t) + u_I(t) \tag{7}$$

$$\begin{aligned} u_I(t) = &-\operatorname{sgn}(\rho^*)\Big(\sum_{l=0;\, l\neq 1}^{p} \gamma_{1l}\, |e(t)|^l \int_0^t e(t)\, |e(t)|^l\, dt \\ &+ \sum_{l=2}^{p} \gamma_{2l}\, |e(t)|^{2l-1} \int_0^t e(t)\, |e(t)|^{2l-1}\, dt \Big) \end{aligned}$$

$$\dot{\theta}(t) = -\chi(t) - \dot{\chi}(t) - \dot{\chi}(t-h), \quad \chi(t) = \operatorname{sgn}(\rho^*)\Gamma^{-1}\omega(t)e(t) \tag{8}$$

where the adaptation gain is $\theta(t) = [\theta_e(t)\ \theta_1^T(t)\ \theta_2^T(t)]^T \in \mathbb{R}^{2n-1}$, and the "regressor" is $\omega(t) = [e(t)\ x_1^T(t)\ x_2^T(t)]^T \in \mathbb{R}^{2n-1}$. $x_1 \in \mathbb{R}^{n-1}$ and $x_2 \in \mathbb{R}^{n-1}$ are the states of the auxiliary filters

$$\begin{aligned} \dot{x}_1(t) &= A_f x_1(t) + b_f u(t),\ x_1(0) = 0 \\ \dot{x}_2(t) &= A_f x_2(t) + b_f y(t),\ x_2(0) = 0 \end{aligned} \tag{9}$$

where $A_f \in \mathbb{R}^{(n-1)\times(n-1)}$ is Hurwitz and b_f is a constant vector such that (A_f, b_f) is controllable. The positive scalars $\gamma_{1l} > 0$, $\gamma_{2l} > 0, l = 1,\ldots,m$, $h > 0$ and the matrix $\Gamma = \Gamma^T > 0$ are tuning parameters, to be chosen by the designer such that the transient response of the controlled system becomes pleasing. Unfortunately, as in most of existing adaptive control schemes, there do not exist systematic and analytic tuning rules for the design parameters. As usual, one has to resort to tuning by trial-and-error.

Remark 2: When comparing the control $u(t)$ with our previous work [12], we notice the following *important difference*: here the control signal $u(t)$ contains the new continuous integral term $u_I(t)$, while in [12] the control signal has an integral term multiplied by a *discontinuous* component.

Remark 3: For adaptation a proportional-integral-time delay (PITD) type algorithm is chosen which may have better transient adaptation performance than the traditional I and PI schemes; for details see e.g. [20], [12].

IV. Error Equation

By using the conventional technique of model reference adaptive control [18], [19] the tracking error $e(t) = y(t) - y_r(t)$ for any $u(t)$ can be expressed as

$$\begin{aligned} e = &W_r(s)\rho^* \Big[u - \theta_e^* y - \theta_1^{*T} x_1 - \theta_2^{*T} x_2 - \theta_r^* r + H(s)d(t) \\ &+ H(s) f(y, y(t-\tau_y(t)), t) + a_\tau^{*T} x(t-\tau_x(t)) \\ &- \theta_1^{*T} H_f(s) a_\tau^{*T} x(t-\tau_x(t)) \Big] \end{aligned} \tag{10}$$

where $\rho^* = \theta_r^{*-1} = k_p k_r^{-1}$, $\theta_1^*, \theta_2^* \in \mathbb{R}^{n-1}$, $\theta_e^* \in \mathbb{R}$, $\theta_r^* \in \mathbb{R}$ are so-called matching parameters, $H(s) = 1 - \theta_1^{*T} H_f(s)$ with $H_f(s) = (sI - A_f)^{-1} b_f = \frac{[s^{n-2}\ \ldots\ s\ 1]^T}{\lambda(s)} \in \mathbb{R}^{n-1}$, $\lambda(s) = s^{n-1} + \cdots + \lambda_1 s + \lambda_0$ is a monic Hurwitz polynomial and A_f, b_f are from (9).

To find a suitable error equation parametrization, we manipulate the last term of (10). Firstly, we introduce a new dynamical system

$$z(t) = \theta_1^{*T} H_f(s)[a_\tau^{*T} x(t-\tau_x(t))] = \theta_z^{*T} z_x(t) \tag{11}$$

where $\theta_z^{*T} = [\theta_{11}^* a_\tau^{*T},\ \theta_{12}^* a_\tau^{*T},\ \ldots,\ \theta_{1(n-1)}^* a_\tau^{*T}]$ and

$$z_x(t) = H_n(s)[x(t-\tau_x(t))] \tag{12}$$

$$H_n(s) = \frac{[I_{n\times n} s^{n-2},\ \ldots,\ I_{n\times n} s,\ I_{n\times n}]^T}{\lambda(s)} \tag{13}$$

Here $\theta_z^* \in \mathbb{R}^{n(n-1)}$, $z_x \in \mathbb{R}^{n(n-1)}$, $H_n(s) \in \mathbb{R}^{n(n-1)\times n}$ and $I_{n\times n}$ is the $n\times n$ identity matrix.

Secondly we decompose the signals $z_{xj}(t)$ in (12) into two components $z_x(t) = z_e(t) + z_r(t)$ where

$$\begin{aligned} z_e(t) &= H_n(s)\big[e_x(t-\tau_x(t))\big] \quad z_r(t) = H_n(s)\big[x_r(t-\tau_x(t))\big] \\ e_x(t-\tau_x(t)) &= x(t-\tau_x(t)) - x_r(t-\tau_x(t)) \end{aligned} \tag{14}$$

where $x_r(t) \in \mathbb{R}^n$ is the state of the reference model (3) with the state space triple (A_r, b_r, c_r).

Then, using (11) and (14), we obtain from (10) the *basic error equation*

$$\begin{aligned} e = &W_r(s)\rho^* \Big[u - \theta_e^* e(t) - \theta_1^{*T} x_1(t) - \theta_2^{*T} x_2(t) - \theta_r^* r(t) - \theta_e^* y_r(t) \\ &+ a_\tau^{*T} x_r(t-\tau_x(t)) - \theta_z^{*T} z_r(t) + a_\tau^{*T} e_x(t-\tau_x(t)) - \theta_z^{*T} z_e(t) \\ &+ H(s)[d(t)] + H(s)[f(y, y(t-\tau_y(t)), t)] \Big] \end{aligned} \tag{15}$$

Remark 4: Note that $e_x(t-\tau_x(t))$ and $z_e(t)$ are not available for measurement, and we shall use them only for analysis, and not for the implementation.

Introducing the parameter error $\tilde{\theta}(t)$ and using the adaptive control (7), the basic tracking error equation (15) can be expressed as

$$\begin{aligned} e =& W_r(s)\rho^*\Big[\tilde{\theta}^T(t)\omega(t)+u_I(t)-\theta_r^* r(t)-\theta_e^* y_r(t)+a_\tau^{*T}x_r(t-\tau_x(t)) \\ & -\theta_z^{*T}z_r(t)+a_\tau^{*T}e_x(t-\tau_x(t))-\theta_z^{*T}z_e(t)+H(s)[d(t)] \\ & +H(s)[f(y,y(t-\tau_y(t)),t)]\Big] \end{aligned} \tag{16}$$

where the parameter error $\tilde{\theta}(t)$ is $\tilde{\theta}(t)=\theta(t)-\theta^*$, and $\theta^*=[\theta_e^*\ \theta_1^{*T}\ \theta_2^{*T}]^T$.

V. STABILITY PROOF

To prove stability, the usual way of MRAC for delay free systems is used, see e.g. [18]. The augmented vector $\hat{x}(t)=[x\ x_1\ x_2]^T$ is introduced, and the state of the corresponding non-minimal realization $\hat{c}^T(sI-\hat{A})^{-1}\hat{b}\theta_r^*$ of W_r is denoted by $\hat{x}_r(t)$. Then we can write the following state space representation for (16)

$$\begin{aligned} \frac{d\hat{e}(t)}{dt} =& \hat{A}\hat{e}(t)+\bar{b}\rho^*\Big\{\tilde{\theta}^T(t)\omega(t)+a_\tau^{*T}\hat{l}^T\hat{e}(t-\tau_x(t))-\theta_z^{*T}C_e\hat{z}_e(t) \\ & -\theta_r^* r(t)-\theta_e^* y_r(t)+u_I(t)+a_\tau^{*T}x_r(t-\tau_x(t))+H(s)[d(t)] \\ & -\theta_z^{*T}H_n(s)\left[x_r(t-\tau_x(t))\right]+H(s)[f(y,y(t-\tau_y(t)),t)]\Big\} \\ \frac{d\hat{z}_e(t)}{dt} =& A_e\hat{z}_e(t)+B_e\hat{l}^T\hat{e}(t-\tau_x(t)),\ z_e(t)=C_e\hat{z}_e(t) \\ e(t) =& y(t)-y_r(t)=\hat{c}^T\hat{e}(t),\ \hat{l}=[I_{n\times n}\ 0_{n\times(n-1)}\ 0_{n\times(n-1)}]^T \end{aligned} \tag{17}$$

where the triple (A_e,B_e,C_e) is a minimal state space realization of the stable transfer matrix $H_n(s)$ in (14), $\bar{b}=\hat{b}\theta_r^*$, and $0_{n\times(n-1)}$ is the zero $n\times(n-1)$ matrix.

Because $\hat{c}^T(sI-\hat{A})^{-1}\hat{b}\theta_r^*=W_r$ is SPR, there exists a matrix $P=P^T>0$ satisfying

$$\hat{A}^TP+P\hat{A}+\zeta\zeta^T+\nu Q=0,\quad P\hat{b}\theta_r^*=P\bar{b}=\hat{c}, \tag{18}$$

where ζ is a vector, $Q=Q^T>0$ is any positive definite matrix and $\nu>0$ is some selective constant, all of which is implied by the Meyer-Kalman-Yakubovich lemma [18, pp.129-130]. Since A_e in (17) is stable, it also holds that

$$A_e^TP_z+P_zA_e+Q_z=0 \tag{19}$$

where $P_z=P_z^T>0$ and $Q_z=Q_z^T>0$.

For the stability analysis we use the following Lyapunov-Krasovskii type functional

$$\begin{aligned} V =& \sum_i^4 V_i,\ V_1=\hat{e}^TP\hat{e}+\hat{z}_e^TP_z\hat{z}_e \\ V_2 =& \tfrac{\nu}{2}\int_{t-\tau_x(t)}^t \hat{e}^T(s)Q\hat{e}(s)ds+\sum_{l=1}^p 2|\rho^*|\varsigma_l\int_{t-\tau_y(t)}^t\left|\hat{e}^T(s)P\bar{b}\right|^{2l}ds \\ V_3 =& |\rho^*|\left(\tilde{\chi}^T(t)\Gamma\tilde{\chi}(t)+\int_{t-h}^t\chi^T(s)\Gamma\chi(s)ds\right) \\ V_4 =& |\rho^*|\sum_{l=0,\,l\neq1}^p\gamma_{1l}^{-1}\left(\theta_{1l}^{vrt}(t)+\xi_{1l}\mathrm{sgn}(e(t))\right)^2 \\ & +|\rho^*|\sum_{l=2}^p\gamma_{2l}^{-1}\left(\theta_{2l}^{vrt}(t)+\varsigma_l\mathrm{sgn}(e(t))\right)^2 \end{aligned} \tag{20}$$

where $\Gamma>0$ is from (8) and

$$\tilde{\chi}(t)=\tilde{\theta}(t)+\chi_0+\chi(t)+\chi(t-h)+\theta_0 \tag{21}$$

The signum function $\mathrm{sgn}(e(t)$ is defined such that $\mathrm{sgn}(e(t))=1$, if $e(t)>0$; $\mathrm{sgn}(e(t))=0$, if $e(t)=0$; and $\mathrm{sgn}(e(t))=-1$, if $e(t)<0$.

The vectors

$$\chi_0=[\frac{r_0}{2\rho^*},\ 0,\ldots,\ 0]^T,\quad \theta_0=[\mathrm{sgn}(\rho^*)(\xi_{11}+\varsigma_1),\ 0,\ldots,\ 0]^T \tag{22}$$

have the same dimension as θ. The "virtual" scalar adaptation gains $\theta_{1l}^{vrt}(t)$ and $\theta_{2l}^{vrt}(t)$, the parameters $r_0>0$, $\varsigma_l>0$ and the positive constants $\xi_{1l}>0, l=1,\ldots,p$ will be defined later.

Using (18) and (19), the time derivatives $\dot{V}_i(t), i=1,\ldots,5$ of (20) along (17) can be written

$$\begin{aligned} \dot{V}_1(t)|_{(17)} =& -\nu\hat{e}^T(t)Q\hat{e}(t)-\hat{e}^T(t)\zeta\zeta^T\hat{e}(t)-\hat{z}_e^T(t)Q_z\hat{z}_e(t) \\ & +2\hat{e}^T(t)P\bar{b}\rho^*a_\tau^{*T}\hat{l}^T\hat{e}(t-\tau_x(t))+2\rho^*\hat{e}^T(t)P\bar{b}\tilde{\theta}^T(t)\omega(t) \\ & -2\hat{e}^T(t)P\bar{b}\rho^*\theta_z^{*T}C_e\hat{z}_e(t)+2\hat{z}_e^T(t)P_zB_e\hat{l}^T\hat{e}(t-\tau_x(t)) \\ & +2\hat{e}^T(t)P\bar{b}\rho^*u_I(t)-2\hat{e}^T(t)P\bar{b}\rho^*\theta_r^*r(t) \\ & -2\hat{e}^T(t)P\bar{b}\rho^*\theta_e^*y_r(t)+2\hat{e}^T(t)P\bar{b}\rho^*a_\tau^{*T}x_r(t-\tau_x(t)) \\ & -2\hat{e}^T(t)P\bar{b}\rho^*\theta_z^{*T}H_n(s)\left[x_r(t-\tau_x(t))\right] \\ & +2\hat{e}^T(t)P\bar{b}\rho^*H(s)[d(t)] \\ & +2\hat{e}^T(t)P\bar{b}\rho^*H(s)[f(y,y(t-\tau_y(t)),t)] \end{aligned} \tag{23}$$

In view the known fact that for any vectors x,y, and any positive-definite matrix S of appropriate dimensions, it holds that $2x^Ty\leq x^TSx+y^TS^{-1}y$ by which we can estimate some of the terms in (23) as follows

$$\begin{aligned} 2\hat{e}^T(t)P\bar{b}\rho^*a_\tau^{*T}\hat{l}^T\hat{e}(t-\tau_x(t)) \leq& \hat{e}^T(t)P\bar{b}\Psi_1\bar{b}^TP\hat{e}^T(t) \\ & +\hat{e}^T(t-\tau_x(t))S\hat{e}(t-\tau_x(t)) \\ -2\hat{e}^T(t)P\bar{b}\rho^*\theta_z^{*T}C_e\hat{z}_e(t) \leq& \hat{e}^T(t)P\bar{b}\Psi_2\bar{b}^TP\hat{e}(t)+\hat{z}_e^T(t)S\hat{z}_e(t) \\ 2\hat{z}_e^T(t)P_zB_e\hat{l}^T\hat{e}(t-\tau_x(t)) \leq& \hat{e}^T(t-\tau_x(t))\Psi_3\hat{e}(t-\tau_x(t)) \\ & +\hat{z}_e^T(t)S\hat{z}_e(t) \end{aligned} \tag{24}$$

where

$$\begin{aligned} \Psi_1 =& \rho^{*2}a_\tau^{*T}\hat{l}^TS^{-1}\hat{l}a_\tau^*,\quad \Psi_2=\rho^{*2}\theta_z^{*T}C_eS^{-1}C_e^T\theta_z^*, \\ \Psi_3 =& \hat{l}B_e^TP_z^TS^{-1}P_zB_e\hat{l}^T \end{aligned} \tag{25}$$

Using boundedness of the reference signals ($|r(t)|\leq r^*$, $|y_r(t)|\leq y_r^*$, $\|x_r(t)\|\leq x_r^*$), and the stability of the transfer functions $H_f(s)$ and $H_n(s)$, in view of (6) we can write the following estimates for the other terms of (23):

$$\begin{aligned} \Xi_{eq} =& -2\hat{e}^T(t)P\bar{b}\rho^*\theta_r^*r(t)-2\hat{e}^T(t)P\bar{b}\rho^*\theta_e^*y_r(t) \\ & +2\hat{e}^T(t)P\bar{b}\rho^*a_\tau^{*T}x_r(t-\tau_x(t))+2\hat{e}^T(t)P\bar{b}\rho^*H(s)[d(t)] \\ & -2\hat{e}^T(t)P\bar{b}\rho^*\theta_z^{*T}H_n(s)\left[x_r(t-\tau_x(t))\right] \\ \leq& 2|e(t)||\rho^*|\Big(|\theta_r^*|r^*+|\theta_e^*|y_r^*+\|a_\tau^*\|x_r^*+\left\|\theta_z^*\right\|\|H_n(s)\|_\infty x_r^* \\ & +\|H(s)\|_\infty d^*\Big) \\ \leq& 2|e(t)||\rho^*|\eta_{eq} \end{aligned} \tag{26}$$

where $\eta_{eq}=|\theta_r^*|r^*+|\theta_e^*|y_r^*+\|a_\tau^*\|x_r^*+\left\|\theta_z^*\right\|\|H_n(s)\|_\infty x_r^*+\|H(s)\|_\infty d^*$.

For the term $\Xi_f=2\hat{e}^T(t)P\bar{b}\rho^*H(s)[f(\star)]$ we have

$$\begin{aligned} \Xi_f \leq& 2|e(t)||\rho^*|\|H(s)\|_\infty\sum_{l=1}^p\Big(\xi_{1l}^*|e(t)|^l+\xi_{2l}^*|e(t-\tau_y(t))|^l \\ & +\xi_{1l}^*|y_r(t)|^l+\xi_{2l}^*|y_r(t-\tau_y(t))|^l\Big) \\ \leq& 2|\rho^*|\|H(s)\|_\infty\sum_{l=1}^p\left(\xi_{1l}^*|e(t)|^{l+1}+\xi_{2l}^*|e(t)||e(t-\tau_y(t))|^l\right) \\ & +\xi_{1l}^*|y_r^*|^l+\xi_{2l}^*|y_r^*|^l\Big) \end{aligned} \tag{27}$$

In view of the inequality $|e(t)|\,|e(t-\tau_y(t))|^l \le \frac{1}{2}|e(t)|^2 + \frac{1}{2}|e(t-\tau_y(t))|^{2l}$ we obtain from (26) and (27)

$$\Xi_{eq}+\Xi_f \le 2\sum_{l=0}^{p}|\rho^*|\,\xi_{1l}|e(t)|^{l+1}+2\sum_{l=1}^{p}|\rho^*|\,\xi_{2l}|e(t-\tau_y(t))|^{2l} \quad (28)$$

where the unknown positive constants ξ_{1l} and ξ_{2l} are $\xi_{10} = \frac{1}{p}\eta_{eq} + \|H(s)\|_\infty |y_r^*|^l(\xi_{10}^* + \xi_{10}^*)$, $\xi_{11} = \|H(s)\|_\infty(\xi_{11}^* + \frac{1}{2}\xi_{21}^*)$ and $\xi_{1l} = \|H(s)\|_\infty \xi_{1l}^*$, $l = 2,\ldots,p$

Substituting (24) and (28) into (23) leads to

$$\begin{aligned}\dot V_1(t)|_{(17)} \le &-\nu\hat e^T(t)Q\hat e(t) - \hat e^T(t)\zeta\zeta^T\hat e(t) - \hat z_e^T(t)(Q_z-2S)\hat z_e(t)\\ &+\hat e^T(t-\tau_x(t))(S+\Psi_3)\hat e(t-\tau_x(t))\\ &+\hat e^T(t)P\bar b(\Psi_1+\Psi_2)\bar b^T P\hat e^T(t) + 2\rho^*\hat e^T(t)P\bar b\tilde\theta^T(t)\omega(t)\\ &+2\sum_{l=0}^{p}|\rho^*|\,\xi_{1l}|e(t)|^{l+1}+2\sum_{l=1}^{p}|\rho^*|\,\xi_{2l}|e(t-\tau_y(t))|^{2l}\\ &+2\hat e^T(t)P\bar b\rho^* u_I(t) \end{aligned} \quad (29)$$

Using (2) for the time derivative $\dot V_2(t)$ of (20) we obtain

$$\begin{aligned}\dot V_2(t)|_{(17)} \le &\tfrac{\nu}{2}\hat e^T(t)Q\hat e(t) - \tfrac{\bar\nu}{2}\hat e^T(t-\tau_x(t))Q\hat e(t-\tau_x(t))\\ &-\sum_{l=1}^{p}2\varsigma_l^*|\rho^*|\left|e(t-\tau_y(t))\right|^{2l}\\ &+2|\rho^*|\,\varsigma_1|e(t)|^2+\sum_{l=2}^{p}2\varsigma_l|\rho^*|\,|e(t)|^{2l}\end{aligned} \quad (30)$$

where $\bar\nu = \frac{\nu\bar\tau_x}{2}$, $\bar\tau_x = 1-\tau_x^* > 0$ and $\varsigma_l^* = (1-\tau_y^*)\varsigma_l$.

To obtain the time derivative $\dot V_3(t)$, we at first note that, in view of (8), the time derivative of $\tilde\chi(t)$ from (21) is

$$\dot{\tilde\chi}(t) = \dot{\tilde\theta}(t)+\dot\chi(t)+\dot\chi(t-h) = -\chi(t) \quad (31)$$

Then, using the last expression and (21), we have

$$\begin{aligned}\dot V_3(t)|_{(17)} =&-2|\rho^*|\,\tilde\chi^T(t)\Gamma\chi(t)-|\rho^*|\,\chi^T(t-h)\Gamma\chi(t-h)\\ &+|\rho^*|\,\chi^T(t)\Gamma\chi(t)\\ =&-|\rho^*|\left[\chi(t)+\chi(t-h)\right]^T\Gamma\left[\chi(t)+\chi(t-h)\right]\\ &-2|\rho^*|\,\tilde\theta^T(t)\Gamma\chi(t)-2|\rho^*|\,\chi_0^T\Gamma\chi(t)-2|\rho^*|\,\theta_0^T\Gamma\chi(t)\end{aligned} \quad (32)$$

Using (22), (8) and (18) for the last terms in (32) we can write

$$\begin{aligned}-2|\rho^*|\,\chi_0^T\Gamma\chi(t) &= -2\rho^*\chi_0^T\omega(t)e(t) = -r_0\hat e^T(t)P\overline{b}\overline{b}^T P\hat e(t)\\ -2|\rho^*|\,\theta_0^T\Gamma\chi(t) &= -2|\rho^*|\,(\xi_{11}+\varsigma_1)e^2(t)\end{aligned} \quad (33)$$

Here, we used the fact that $e(t) = \hat c^T\hat e(t)$, see (17). Substituting (33) in (32) we obtain

$$\begin{aligned}\dot V_3(t)|_{(17)} \le &-2|\rho^*|\,\tilde\theta^T(t)\Gamma\chi(t) - r_0\hat e^T(t)P\overline{b}\overline{b}^T P\hat e(t)\\ &-2|\rho^*|\,(\xi_{11}+\varsigma_1)e^2(t)\end{aligned} \quad (34)$$

Applying (24), (26) and (27) to (23), and using (2) and (34) yields the next equation for the time derivative of $V_\Sigma = V_1+V_2+V_3$,

$$\begin{aligned}\dot V_\Sigma|_{(17)} \le &-\hat e^T(t)\tfrac{\nu}{2}Q\hat e(t)-\hat e^T(t)\zeta\zeta^T\hat e(t)-\hat z_e^T(t)\Big(Q_z-2S\Big)\hat z_e(t)\\ &-2\sum_{l=1}^{p}|\rho^*|\,\big(\varsigma_l-\xi_{2l}\big)|e(t-\tau_y(t))|^{2l}+\sum_{l=2}^{p}2\varsigma_l|\rho^*|\,|e(t)|^{2l}\\ &-\hat e^T(t)P\bar b\Big(r_0 I-\Psi_1-\Psi_2\Big)\bar b^T P\hat e(t)\\ &-\hat e^T(t-\tau_x(t))\Big(\bar\nu Q-S-\Psi_3\Big)\hat e(t-\tau_x(t))\\ &+2\sum_{l=0}^{p}|\rho^*|\,\xi_{1l}|e(t)|^{l+1}+2e(t)\rho^* u_I(t)\end{aligned} \quad (35)$$

For convenience, let us define $Q = Q_1+Q_2$ and $Q_z = Q_{z1}+Q_{z2}$ with $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $Q_{z1} = Q_{z1}^T > 0$ and $Q_{z2} = Q_{z2}^T > 0$, and select values for r_0, Q_2, Q_{z2} and ς_l of (30) from the inequalities

$$\begin{aligned}r_0 > \lambda_{max}(\Psi_1+\Psi_2),\ &\lambda_{min}(\bar\nu Q_2) > \lambda_{max}(S+\Psi_3),\\ \lambda_{min}(Q_{z2}) > \lambda_{max}(2S),\ &\varsigma_l > \xi_{2l}\end{aligned} \quad (36)$$

Remark 5: We note that the coefficient matrices Q, Q_z and S and the scalar parameters r_0 and ς_l are used only for analysis and do not influence the control law. Controller gains adjust automatically to counter the non-desirable effects of delayed states, parameter uncertainties and an external disturbance.

Then, in view of (36) we obtain from (35)

$$\begin{aligned}\dot V_\Sigma|_{(17)} \le &-\hat e^T(t)\tfrac{\nu}{2}Q_1\hat e(t)-\hat e^T(t)\zeta\zeta^T\hat e(t)-\hat z_e^T(t)Q_{z1}\hat z_e(t)\\ &-\hat e^T(t-\tau_x(t))\bar\nu Q_1\hat e(t-\tau_x(t))+2\rho^* e(t)u_I(t)\\ &+2\sum_{l=0,\,l\ne1}^{p}|\rho^*|\,\xi_{1l}|e(t)|^{l+1}+\sum_{l=2}^{p}2\varsigma_l|\rho^*|\,|e(t)|^{2l}\end{aligned} \quad (37)$$

Let us define the "virtual" adaptation gains $\theta_{1l}^{vrt}(t)$ and $\theta_{2l}^{vrt}(t)$ in (20) as

$$\begin{aligned}\dot\theta_{1l}^{vrt}(t) &= -\gamma_{1l}e(t)\,|e(t)|^l, \quad \theta_{1l}^{vrt}(0)=0;\\ \dot\theta_{2l}^{vrt}(t) &= -\gamma_{2l}e(t)\,|e(t)|^{2l-1}, \quad \theta_{2l}^{vrt}(0)=0,\ l=1,\ldots,p.\end{aligned} \quad (38)$$

Now in view of (38) the time derivative $\dot V_4(t)$ is

$$\begin{aligned}\dot V_4(t)|_{(17)}^{e(t)\ne0} =&2|\rho^*|\sum_{l=0,\,l\ne1}^{p}\gamma_{1l}^{-1}\big(\theta_{1l}^{vrt}(t)+\mathrm{sgn}(e(t))\xi_{1l}\big)\dot\theta_{1l}^{vrt}(t)\\ &+2|\rho^*|\sum_{l=2}^{p}\gamma_{2l}^{-1}\big(\theta_{2l}^{vrt}(t)+\mathrm{sgn}(e(t))\varsigma_l\big)\dot\theta_{2l}^{vrt}(t)\\ =&2|\rho^*|\sum_{l=0,\,l\ne1}^{p}\Big(\gamma_{1l}e(t)\,|e(t)|^l\int_0^t e(t)\,|e(t)|^l\\ &-\xi_{1l}\,|e(t)|^{l+1}\Big)+2|\rho^*|\sum_{l=2}^{p}\Big(\gamma_{2l}e(t)\,|e(t)|^{2l-1}\times\\ &\times\int_0^t e(t)\,|e(t)|^{2l-1}-\varsigma_l\,|e(t)|^{2l}\Big)\\ \dot V_4(t)|_{(17)}^{e(t)=0} =&0\end{aligned} \quad (39)$$

Substituting $u_I(t)$ from (7) in (37) and using (37) and (39) we obtain for $\dot V$ in (20)

$$\begin{aligned}\dot V|_{(17)} \le &-\hat e^T(t)\tfrac{\nu}{2}Q_1\hat e(t)-\hat e^T(t)\zeta\zeta^T\hat e(t)-\hat z_e^T(t)Q_{z1}\hat z_e(t)\\ &-\hat e^T(t-\tau_x(t))\bar\nu Q_1\hat e(t-\tau_x(t))\end{aligned} \quad (40)$$

This implies, e.g. [18], [19] that V and, therefore, $\hat e(t)$, $e(t)$, $\hat z_e(t)$, $\theta_{1l}^{vrt}(t)$, $\theta_{2l}^{vrt}(t)$, $\tilde\theta$, $\theta \in L_\infty$. This fact is central to the remainder of the stability analysis, which follows directly using the steps in [18].

We summarize the main result as

Theorem 1: Consider the closed-loop system defined by the plant in (1), the controller in (7), and the updating algorithm in (8) with the reference model as in (3). Then for the bounded disturbance $d(t)$ and the nonlinearity $f(\cdot)$ satisfying the inequality (6), and for any delays $\tau_x(t)$ and $\tau_y(t)$ that satisfy (2), the following two properties hold: (*i*) all signals of the closed-loop system are bounded and (*ii*) $\lim_{t\to\infty} e(t) = 0$.

Remark 6: Theorem 1 shows that the stability of the closed-loop system and the controller parameters are completely *independent* of the value of the plant time-delays $\tau_x(t)$ and $\tau_y(t)$. The controller is also robust to an external disturbance.

Remark 7: It can be shown that the results are valid for the more general system with multiple time-varying delays:

$$\begin{aligned}\dot{x}(t) =& Ax(t) + \sum_{j=1}^{M} A_{\tau j} x(t-\tau_{xj}(t)) + bu(t) + bd(t)\\ & + bf(y(t), y(t-\tau_{y1}(t)), \ldots, y(t-\tau_{yN}(t)), t)\\ y(t) =& c^T x(t) \end{aligned} \quad (41)$$

VI. Simulation Results

We illustrate the results by a nonlinear unstable second order time delay system from (1) with

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \; A_\tau = \begin{bmatrix} -0.2 & -0.1 \\ 0 & 0 \end{bmatrix},$$
$$A_r = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, \; b_r = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$c = [0.5 \;\; 0.5]^T$, $x(0) = [1 \;\; 1]^T$, $x_r(0) = [0 \;\; 0]^T$, $c_r = [0.5 \;\; 0.5]^T$, $d(t) = 0.2sin(0.2t)$ and $f(.) = y^3(t-\tau_y(t)) + y^2(t)$.

All plant parameters, the nonlinearity $f(.) = y^3(t-\tau_y(t)) + y^2(t)$ and the disturbance $d(t) = 0.2sin(0.2t)$ are unknown to the controller. The component $u_I(t)$ from (7) is

$$\begin{aligned} u_I(t) =& -\gamma_{10} \int_0^t e(t)dt - \gamma_{12}\,|e(t)|^2 \int_0^t e(t)\,|e(t)|^2\, dt \\ & -\gamma_{23}\,|e(t)|^5 \int_0^t e(t)\,|e(t)|^5\, dt \end{aligned} \quad (42)$$

The parameter values of the adaptive controller (7) in our simulation were chosen as $\gamma_{10} = 10$, $\gamma_{12} = 10$, $\gamma_{23} = 10$ $A_f = -1$ and $b_f = 1$. The gain $\theta(t)$ in (8) is calculated as

$$\frac{d\theta(t)}{dt} = -30\omega(t)e(t)dt - 10\frac{d\omega(t)e(t)}{dt} - 2\frac{d\omega(t-1)e(t-1)}{dt}.$$

The command signal $r(t)$ is a square signal with amplitude 0.9 and basic frequency 0.2 rad/s.

Simulation results are shown in Figures 1–2, where we show the time responses of the output of the plant $y(t)$, the output of the reference model $y_r(t)$, the tracking error $e(t)$, the control signal $u(t)$ and the adaptation gain vector $\theta(t)$. To illustrate the robustness of the adaptive control law to delay uncertainty we considered two cases of the delay values $\tau_x = 5$, $\tau_y = 2$ and $\tau_x(t) = 4.5 + 3sin(t), \tau_y(t) = 4.5 - 3sin(t/2)$ in the plant model. The controller parameters remained the same for the both cases. Note, that in this example for the time-varying delay case, $\dot{\tau}_x(t)$ and $\dot{\tau}_y(t)$ do not satisfy the restrictive conditions (2).

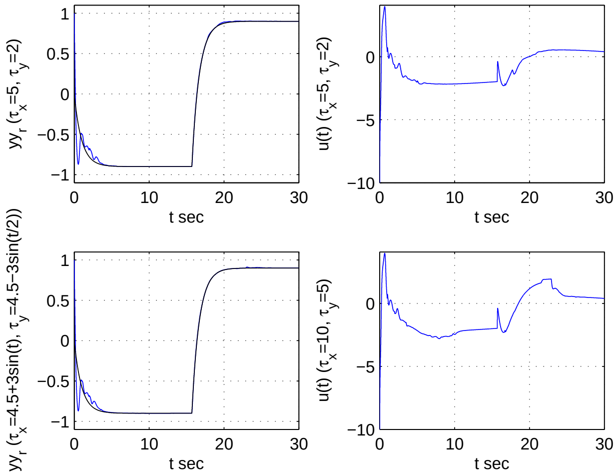

Fig. 1. Simulation of the adaptive control system for the nonlinear plant (1) and the controller (7). The left and right graphs show the time history of the plant and reference model outputs y_t, $y_r(t)$, and the control signal $u(t)$, respectively for the two cases of the time delays values.

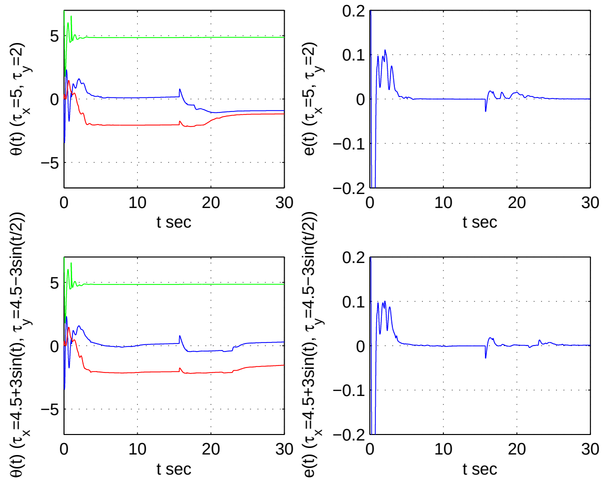

Fig. 2. Simulation of the adaptive control system for the nonlinear plant (1) and the controller (7). The left and right graphs show the time history of the vector adaptation gain $\theta(t)$ and the tracking error $e(t)$, respectively for the two cases of the time delays values.

VII. Concluding Remarks

An effective continuous adaptive controller, based on a new error equation parametrization, is proposed to achieve tracking of reference signals with asymptotical zero error for a class of nonlinear plants. To achieve robust properties both with respect to unknown time-varying plant delays, and to an external disturbance with unknown bounds, we introduce a new integral control component with constant gain. For the considered class of nonlinear delayed systems, the proposed adaptive control law constructions make economical use of known results of model reference adaptive control. Both for the stability analysis and synthesis, we use a special form Lyapunov-Krasovskii type functional which includes a term with special "virtual" adaptation gains.

We hope that the proposed way to design can be used to solve adaptive control problem for an another class of plants than those described by (1), e.g. for

$$\begin{aligned} f\big(y(t), y(t-\tau_y(t)), t\big) \leq & \sum_{l=1}^{p} \xi_{1l}^{*} |y(t) z_l(y(t))| \\ & + \sum_{l=1}^{p} \xi_{2l}^{*} |y(t-\tau_y(t)) z_l(y(t-\tau_y(t))| \end{aligned}$$

where $z_l(\star)$, $(l = 1, \ldots, p)$ are known nonlinear functions and ξ_{1l}^{*} and ξ_{2l}^{*} are unknown positive constants. We are currently working to develop a suitable adaptive control parametrization.

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