Steady-State and N-Stages Control for Isolated Controlled Intersections

Jack Haddad, Bart De Schutter, David Mahalel, and Per-Olof Gutman

***Abstract*— In this paper a simplified isolated controlled intersection is introduced. Discrete-event piecewise affine (PWA) and discrete-event max-plus models are proposed to formulate the optimization problem for the switching sequences. Two control problems are considered: steady-state control and N-stages control. The formulated discrete-event PWA and max-plus problems are converted to be solved by linear programming (LP), mixed-integer programming (MIP), and mixed-integer linear programming (MILP). In the special case when the criterion is a strictly increasing and linear function of the queue lengths, the steady-state control problem is solved analytically.**

I. Introduction

Transportation networks over the world are becoming more and more congested. Congestion has several effects on travelers, businesses, agencies, and cities. One significant element is the value of the additional time and wasted fuel. The congestion in USA's metropolitan areas is increasing continuously, e.g. in 2005 congestion (based on wasted time and fuel) cost about \$ 78.2 billion or an average of \$ 707 per traveler [21].

As traffic becomes more congested, utilization of the available infrastructure and increasing the capacity are an essential goal, which can be achieved by traffic control and management. The urban transportation network consists of number of signalized intersections. The congestion is not distributed equally between all the signalized intersections. There is usually a group of intersections that are more congested than others. These intersections are called critical intersections. Increasing the capacity of critical intersections will increase the total throughput flow of the network, and as a result the network capacity will be increased and the delays will decrease.

Different models, methods, and strategies have been proposed and applied for controlling urban isolated signalized intersections [1], [2], [10], [14], [15], [17], [18], [20], [22]. These researches aim to minimize delays or to maximize the intersection capacity. Some recent research considers the isolated intersection in the urban traffic network as a hybrid system [6], [7], [9], [16] and others propose the game theory approach [23] to model signalized intersections.

In [4], the optimal acyclic (or N-stages) control was dealt with, where the Extended Linear Complementary Problem (ELCP), which is a mathematical programming problem, was used. In this paper, we introduce the steady-state and N-stages control problems and study the design of optimal traffic signal switching time sequences for a traffic signal controlled intersection through discrete-event models: max-plus and PWA (piecewise affine).

The paper is organized as follows. After describing the problem definition in Section II, the discrete-event models of an isolated intersection and the formulation of the optimal problems are given in Section III. The control problem for steady-state and N-stages control is dealt with in Sections IV and V respectively, which is followed by conclusions and topics for future research.

II. Problem Definition

In this paper, a typical simplified isolated intersection will be dealt with[1]. As shown in Fig. 1, there are two movements (m_1 and m_2), where each movement has a traffic signal that can be green or red. There is a traffic conflict in the intersection area between the two movements, therefore they cannot travel simultaneously and the traffic signal will be opposite, i.e. when movement m_1 has a green light movement m_2 has a red light and vice versa.
A given movement will encounter intertwined green and red periods. A cycle (s) is defined as a pair of one green and one red period, and may change over time.

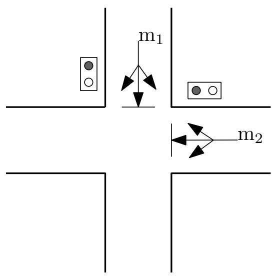

Fig. 1. Simplified isolated controlled intersection

In this case there are two movements m_1 and m_2, therefore the evolution of the queue lengths will be considered only for these two movements. The length of queue movement i at time t, which is the number of vehicles stopping behind the stop line in the intersection, is denoted by $q_i(t)$ (veh). Let $f_{\mathrm{arr},i}(t)$, $f_{\mathrm{dep},i}(t)$ be, respectively, the arrival rate (veh/s) and the departure rate (veh/s) for queue i at time t. The queue length growth rate $\alpha_i(t)$ (veh/s) for queue i at time t is given by $\alpha_i(t) = f_{\mathrm{arr},i}(t) - f_{\mathrm{dep},i}(t)$.

[1]Extension to more complex arrangements or setups is possible.

Jack Haddad, David Mahalel, and Per-Olof Gutman are with the Transportation Research Institute, Technion - Israel Institute of Technology, Haifa 32000, Israel.
Jack Haddad, jh@technion.ac.il
David Mahalel, mahalel@technion.ac.il
Per-Olof Gutman, peo@technion.ac.il
Bart De Schutter is with the Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. b@deschutter.info

The following assumptions are made:

- A1: The arrival and departure rates in the isolated intersection are known and constant within each cycle[2].
- A2: When the traffic signal is green, the departure rate is bigger than the arrival rate, i.e. $f_{\text{dep},i}(t) > f_{\text{arr},i}(t)$, and when the traffic signal is red, the departure rate is equal to zero, $f_{\text{dep},i}(t) = 0$, and the arrival rate is bigger than zero, $f_{\text{arr},i}(t) \geq 0$.
- A3: The queue lengths (number of vehicles) are approximated by real numbers.
- A4: Each movement will have only one green signal per cycle.

For the isolated controlled intersection with a constant traffic arrival and departure rates, we determine the control traffic signals that optimize the given control objective or criterion. We also formulate the stability conditions for the optimal traffic signal solution in the two control cases: steady-state and N-stages control.

III. Discrete Models for Isolated Controlled Intersections

A variety of models [8], [19] are based on the store-and-forward approach of modeling traffic networks that was first suggested by [11], [12]. This approach enables the simplification of the mathematical description of the traffic flow process without the use of switched variables. In this paper we consider the isolated controlled intersection as switching systems, as was done in [3], [4], [5].

Optimizing traffic signal switching sequences will be done through two models: discrete-event max-plus and discrete-event PWA.

A. Basic model

Let k be the index of the cycle. By assumption A4, in the cycle sequence each movement (m_1 or m_2) will have only one green signal per cycle. For each cycle of the cycle sequence we want to determine two decision variables: the cycle time, $T_{\text{cyc},k}$ (s), and g_k (%) the proportion of the green time of movement m_1 in cycle k. The cycle time is expressed as number of seconds and the proportion of the green time is expressed as a percentage of the cycle time.

The evolution of the system begins at time t_0. This implies that the state of the queue length i at time t is given by

$$q_i(t) = q_i(t_0) + \int_{t_0}^{t} \alpha_i(t)dt \tag{1}$$

There are two switching times for cycle k: t_{2k+1} and t_{2k+2} (see Fig. 2). Without loss of generality, let the green light for movement m_1 start at t_{2k}, which coincides with the start of the cycle time. Hence, t_{2k+1} is the end of the green light for movement m_1 (or the start of the green light for movement m_2) and t_{2k+2} is the the end of the green light for movement m_2 (or the start of the green light for movement m_1 in the next cycle time). The cycle time duration $T_{\text{cyc},k}$ is equal to $t_{2k+2} - t_{2k}$. By assumption A1 the arrival rate

[2]Also averaged values can be considered.

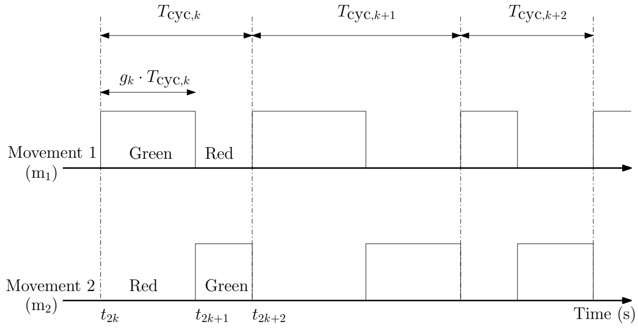

Fig. 2. Traffic signal switching sequences for movements m_1 and m_2

of queue i in phase $2k$ (i.e. the time period between t_{2k} and t_{2k+1}), $f_{\text{arr},i}(t_{2k})$, and the departure rate of queue i in phase $2k$, $f_{\text{dep},i}(t_{2k})$, are known and constant. The same holds for phase $2k+1$: the arrival rate of queue i, $f_{\text{arr},i}(t_{2k+1})$, and the departure rate of queue i, $f_{\text{dep},i}(t_{2k+1})$, are known and constant. This means, for example, that the growth rate $\alpha_i(t_{2k}) = f_{\text{arr},i}(t_{2k}) - f_{\text{dep},i}(t_{2k})$ has a constant rate value between the two discrete event time t_{2k} and t_{2k+1}.
The relations between the time sequences are the following,

$$t_{2k+1} = t_{2k} + g_k \cdot T_{\text{cyc},k} \tag{2}$$

$$t_{2k+2} = t_{2k} + T_{\text{cyc},k} \tag{3}$$

B. Formulation of an optimal discrete-event max-plus problem

The value of the queue length for movement m_1 in cycle k at the switching time instant t_{2k+1} is given by

$$q_1(t_{2k+1}) = \max(q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot g_k \cdot T_{\text{cyc},k}, 0) \tag{4}$$

and at the switching time instant t_{2k+2} is given by

$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + \alpha_1(t_{2k+1}) \cdot (1 - g_k) \cdot T_{\text{cyc},k} \tag{5}$$

Recall that the signal light for movement m_2 is opposite to m_1, therefore the value of the queue lengths for movement m_2 in cycle k are given by

$$q_2(t_{2k+1}) = q_2(t_{2k}) + \alpha_2(t_{2k}) \cdot g_k \cdot T_{\text{cyc},k} \tag{6}$$

$$q_2(t_{2k+2}) = \max(q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot (1 - g_k) \cdot T_{\text{cyc},k}, 0) \tag{7}$$

We now consider the following problem: for a given number of cycles N and starting time t_0 (recall that the starting time should also coincide with the start of green signal for m_1), we compute an optimal switching time sequence $t_1, t_2, \ldots, t_{2N}$ that minimizes a given performance criterion J. There are variety of criteria that can be chosen, e.g. average queue length, maximal queue length, and delay over all queues.
Two new variables are defined $T_1(k)$ (s) and $T_2(k)$ (s), where $T_1(k) = g_k \cdot T_{\text{cyc},k}$ and $T_2(k) = (1-g_k) \cdot T_{\text{cyc},k}$. Substituting these variables into (4) - (7) leads to the following Discrete-event Max-Plus (DMP) problem:

$$\min_{\substack{T_1(0),\,T_2(0),\\ T_1(1),\,T_2(1),\,\cdots,\\ T_1(N-1),\,T_2(N-1)}} J \tag{8}$$

subject to

$$q_1(t_{2k+1}) = \max(q_1(t_{2k}) + \alpha_1(t_{2k}) \cdot T_1(k), 0) \quad (9)$$
$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + \alpha_1(t_{2k+1}) \cdot T_2(k) \quad (10)$$
$$q_2(t_{2k+1}) = q_2(t_{2k}) + \alpha_2(t_{2k}) \cdot T_1(k) \quad (11)$$
$$q_2(t_{2k+2}) = \max(q_2(t_{2k+1}) + \alpha_2(t_{2k+1}) \cdot T_2(k), 0) \quad (12)$$

for $k = 0, 1, 2, \ldots, N-1$.
The optimization problem is formulated by minimization of the criterion J over N cycles. Hence, the number of variables to be determined is $2N$.

IV. Steady-State Control

In order to develop and to implement dynamic models for controlling the traffic system, which can be helpful in decreasing congestion, first the steady-state control problem will be solved. The optimal solution and the feasibility condition for the steady-state control are useful in the control theory for the N-stages control problem, e.g., the steady-state solution can be the initial solution for the optimization process for the N-stages control problem.

In the steady-state control problem it is assumed that after some time, denoted by τ_0 ($k = 0$), the system will be in a steady-state mode. In the steady-state mode the cycle time and the green time will be constant, i.e. the traffic flows at the intersection and the evolution of the queues at the stop lines will be cyclic. Hence, only one cycle and two switching times (τ_1 and τ_2) are required to calculate the optimal cyclic switching sequences in the steady-state mode. The queue length for movement i at the start of the cycle will be equal to the queue length at the start of the next cycle:

$$q_1(\tau_0) = q_1(\tau_2) \quad (13)$$
$$q_2(\tau_0) = q_2(\tau_2) \quad (14)$$

In the following, we consider the case when the criterion J is a strictly increasing[3] function of the queue lengths (i.e. of $q_1(\tau_1)$, $q_2(\tau_1)$, $q_1(\tau_2)$ and $q_2(\tau_2)$), such as average queue length, positively weighted sum of queue lengths, or average travel time. The second case when the criterion J is not a strictly increasing function of the queue lengths, such as maximum queue length, weighted sum of queue lengths with some weights equal to zero, or maximal travel time was dealt with in [13].

A. The criterion is a strictly increasing function of the queue lengths

Now we show that for a criterion that is a strictly increasing function of the queue lengths the optimal cyclic switching sequences problem and the feasibility condition for the steady-state control can be formulated through a discrete-event max-plus model, and solved analytically for a strictly increasing and linear criterion.

[3]In the steady-state control, the queue length vector q is defined as $(q_1(\tau_1), q_2(\tau_1), q_1(\tau_2), q_2(\tau_2))$, then the function J is strictly increasing if for all queue length vectors with $\hat{q} \le \tilde{q}$ and with $\hat{q}_i < \tilde{q}_i$ for at least one index i, we have $J(\hat{q}) < J(\tilde{q})$.

1) Formulation of an optimal cyclic discrete-event max-plus problem: The formulation is based on the DMP problem (8) - (12). The cyclic queue lengths equations (13) - (14) are added to the DMP problem and then we optimize it over only one cycle time ($N = 1$ and $k = 0$). Therefore, the number of decision variables will decrease to two: $T_1(0)$ and $T_2(0)$. For simplicity we write $T_1(0)$ and $T_2(0)$ as T_1 and T_2, respectively. We also assume that a lower bound $T_{\min}$ (with $T_{\min} > 0$) for the sum of T_1 and T_2 is given, i.e. $T_1 + T_2 \ge T_{\min}$.
The Cyclic Discrete-event Max-Plus (CDMP) problem is then defined as follows:

$$\min_{T_1, T_2} J \quad (15)$$

subject to

$$q_1(\tau_1) = \max(q_1(\tau_0) + \alpha_1(\tau_0) \cdot T_1, 0) \quad (16)$$
$$q_1(\tau_2) = q_1(\tau_1) + \alpha_1(\tau_1) \cdot T_2 \quad (17)$$
$$q_2(\tau_1) = q_2(\tau_0) + \alpha_2(\tau_0) \cdot T_1 \quad (18)$$
$$q_2(\tau_2) = \max(q_2(\tau_1) + \alpha_2(\tau_1) \cdot T_2, 0) \quad (19)$$
$$T_1 + T_2 \ge T_{\min} \quad (20)$$
$$\text{and } (13), (14)$$

Note that for scalars $a, b, c \in \mathbb{R}$ we have that $a = \max(b, c)$ implies $a \ge b$ and $a \ge c$. In a similar way the CDMP problem can be rewritten in such a way that the max equations are "relaxed" to linear inequality equations. But first, the cyclic queue lengths equations (13) and (14) are substituted into (16) and (18) respectively,

$$q_1(\tau_1) = \max(q_1(\tau_2) + \alpha_1(\tau_0) \cdot T_1, 0) \quad (21)$$
$$q_2(\tau_1) = q_2(\tau_2) + \alpha_2(\tau_0) \cdot T_1 \quad (22)$$

The max equations (19) and (21) can be relaxed into linear inequality equations as follows,

$$q_1(\tau_1) \ge q_1(\tau_2) + \alpha_1(\tau_0) \cdot T_1 \quad (23)$$
$$q_1(\tau_1) \ge 0 \quad (24)$$
$$q_2(\tau_2) \ge q_2(\tau_1) + \alpha_2(\tau_1) \cdot T_2 \quad (25)$$
$$q_2(\tau_2) \ge 0 \quad (26)$$

This leads to the "Relaxed" Cyclic Discrete-event Max-Plus (R-CDMP) problem:

$$\min_{T_1, T_2} J \quad (27)$$

subject to

$$(17), (20), (22), (23), (24), (25), (26)$$

Proposition 1: If the criterion J is a strictly increasing function of the queue lengths, then any optimal solution of the R-CDMP problem is also an optimal solution of the CDMP problem.

Proof: See proof of Proposition 1 of [13]. ∎

So in the sequel we consider the R-CDMP problem instead of the CDMP problem.

2) Feasibility condition: In this section, the existence condition for the steady-state control is derived based on the R-CDMP problem.
We can eliminate $q_1(\tau_2)$ and $q_2(\tau_1)$ from the constraints of the R-CDMP problem by substituting (17) and (22) into (23) and (25) respectively, resulting in

$$-\alpha_1(\tau_0)\cdot T_1 \geq \alpha_1(\tau_1)\cdot T_2 \tag{28}$$

$$-\alpha_2(\tau_1)\cdot T_2 \geq \alpha_2(\tau_0)\cdot T_1 \tag{29}$$

If $T_1 = 0$, then it is implied from (28) and (29) that $T_2 = 0$, and vice versa. But this is a contradiction with (20) and the fact that $T > 0$. Hence, we will have $T_1 > 0$ and $T_2 > 0$. However, it follows from assumptions A1 and A2 that $\alpha_1(\tau_0) < 0$ and $\alpha_2(\tau_1) < 0$ where $\alpha_1(\tau_1) > 0$ and $\alpha_2(\tau_0) > 0$. Hence, from (28) and (29) we obtain the following feasibility condition:

$$\frac{\alpha_1(\tau_1)}{-\alpha_1(\tau_0)} \leq \frac{-\alpha_2(\tau_1)}{\alpha_2(\tau_0)} \tag{30}$$

or by symmetry

$$\frac{-\alpha_1(\tau_0)}{\alpha_1(\tau_1)} \geq \frac{\alpha_2(\tau_0)}{-\alpha_2(\tau_1)} \tag{31}$$

Proposition 2: The steady-state solution exists for every initial queue lengths if the arrival and departure rates satisfy the feasibility condition (30).

3) Analytic solution for linear criterion: Now we show that if the criterion J is a strictly increasing *linear* function of the queue lengths, then the R-CDMP problem can be solved analytically.

As an example, let the sum of maximum queue lengths be our criterion function J,

$$J = w_1 q_1(\tau_2) + w_2 q_2(\tau_1) \tag{32}$$

where $w_1, w_2 > 0$.
Note that J is strictly increasing in the queue lengths due to the fact that $w_1, w_2 > 0$.

Proposition 3: The optimal solutions of the queue lengths $q_1(\tau_1)$ and $q_2(\tau_2)$ in the R-CDMP problem with the linear criterion (32) must be equal to zero in order to minimize J, i.e.

$$q_1^*(\tau_1) = 0 \tag{33}$$

$$q_2^*(\tau_2) = 0 \tag{34}$$

Proof: The proof is done by contradiction. We first assume that $q_1(\tau_1) > 0$ and $q_2(\tau_2) > 0$, the $\max$ equations (21) and (19) can be written as follows,

$$q_1(\tau_1) = q_1(\tau_2) + \alpha_1(\tau_0)\cdot T_1 \tag{35}$$

$$q_2(\tau_2) = q_2(\tau_1) + \alpha_2(\tau_1)\cdot T_2 \tag{36}$$

This leads to slightly changes in the equations (28) and (29), respectively

$$-\alpha_1(\tau_0)\cdot T_1 = \alpha_1(\tau_1)\cdot T_2 \tag{37}$$

$$-\alpha_2(\tau_1)\cdot T_2 = \alpha_2(\tau_0)\cdot T_1 \tag{38}$$

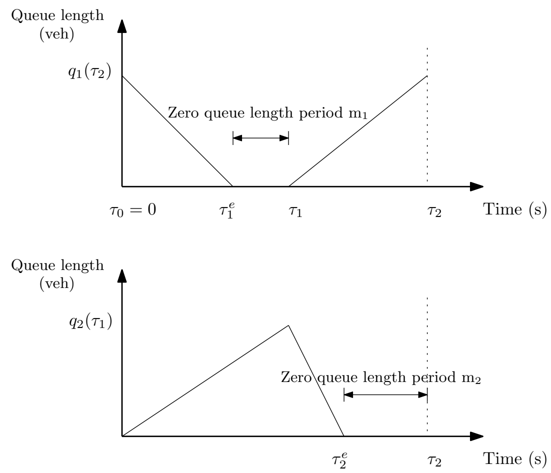

Fig. 3. Zero queue length periods for movement m_1 and m_2

and the R-CDMP problem will be as follows,

$$\min_{T_1, T_2} J \tag{39}$$

subject to

$$(17), (20), (22), (35), (36)$$

Equations (37), (38) and (20) imply that $q_1(\tau_1)$ and $q_2(\tau_2)$ are constant values and not related to the variables T_1 and T_2. Hence, in order to minimize J (32) the constant values must be equal to zeros. ■

We define a "zero queue length period" to the period of time (larger than zero) when the queue length is equal to zero (see Fig. 3). Movement can encounter only one zero queue length period per a cycle, and it may happen after the start of the green light and before the start of the red light, i.e., the time between τ_0 and τ_1 for movement m_1, and the time between τ_1 and τ_2 for movement m_2. Let us denote the start of the zero queue length period for movements m_1 and m_2 by τ_1^e and τ_2^e, respectively. Then the zero queue length period for movement m_1,m_2 starts at time τ_1^e,τ_2^e and ends at time τ_1,τ_2, respectively.
Without losing of generality we fill in $\tau_0 = 0$, then the cycle time T is equal to τ_2.

Proposition 4: The optimal cycle time of the R-CDMP problem with the criterion J, which is a strictly increasing linear function of the queue lengths, is equal to the minimum cycle time $T_{\min}$.

Proof: The general case where the cycle time is bigger than the minimum cycle time and each movement has a zero queue length period is shown in Fig. (4). The cycle time τ_2 can be decreased to $T_{\min}$ by multiplication all the values by the coefficient $\gamma = \frac{T_{\min}}{\tau_2}$ as shown in Fig. (4). Decreasing the cycle time decrease the maximum queue lengths from $q_1(\tau_2)$ and $q_2(\tau_1)$ to $\gamma q_1(\tau_2)$ and $\gamma q_2(\tau_1)$, respectively, and decreases the value of the criterion J, i.e. the maximum queue lengths decreases as we decrease the cycle time, which proves that the optimal cycle time will be equal to the minimum cycle time $T_{\min}$. ■

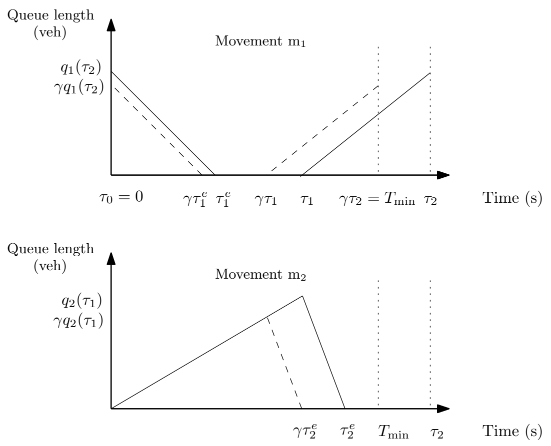

Fig. 4. Decreasing cycle time to the minimum by scaling multiplication

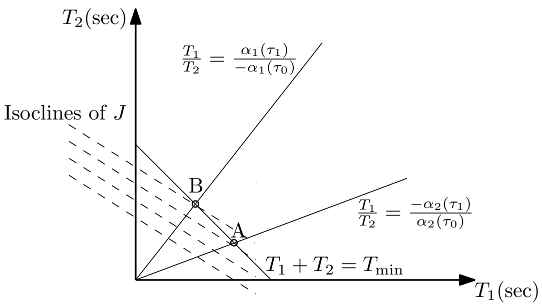

Fig. 5. Analytic solution for the linear programming problem

According to the Propositions (3) and (4), we obtain the following linear programming (LP) problem

$$\min_{T_1,T_2} J = w_2\alpha_2(\tau_0)\cdot T_1 + w_1\alpha_1(\tau_1)\cdot T_2 \tag{40}$$

subject to

$$-\alpha_1(\tau_0)\cdot T_1 \geq \alpha_1(\tau_1)\cdot T_2 \tag{41}$$

$$-\alpha_2(\tau_1)\cdot T_2 \geq \alpha_2(\tau_0)\cdot T_1 \tag{42}$$

$$T_1 + T_2 = T_{\min} \tag{43}$$

In the case when the feasibility condition (Eq. 30) is satisfied by the strictly inequality constraint, i.e. $\frac{\alpha_1(\tau_1)}{-\alpha_1(\tau_0)} < \frac{-\alpha_2(\tau_1)}{\alpha_2(\tau_0)}$, the solution of this problem depends on the slope of the linear objective function (see Fig. 5). If $w_2\alpha_2(\tau_0) < w_1\alpha_1(\tau_1)$ the optimal solution will be point A, where in this point the movement m_2 will not have a zero queue length period. When $w_2\alpha_2(\tau_0) > w_1\alpha_1(\tau_1)$ the optimal solution will be point B, and the movement m_1 will not have a zero queue length period. Points A and B are equal to

$$(T_1,T_2)_A = \left(\frac{-T_{\min}\alpha_2(\tau_1)}{\alpha_2(\tau_0)-\alpha_2(\tau_1)}, \frac{T_{\min}\alpha_2(\tau_0)}{\alpha_2(\tau_0)-\alpha_2(\tau_1)}\right) \tag{44}$$

$$(T_1,T_2)_B = \left(\frac{-T_{\min}\alpha_1(\tau_1)}{\alpha_1(\tau_0)-\alpha_1(\tau_1)}, \frac{T_{\min}\alpha_1(\tau_0)}{\alpha_1(\tau_0)-\alpha_1(\tau_1)}\right) \tag{45}$$

If $w_2\alpha_2(\tau_0) = w_1\alpha_1(\tau_1)$ all points between A and B (i.e. the combination of $\alpha\,(T_1,T_2)_A + (1-\alpha)\,(T_1,T_2)_B$ where $\alpha \leq 1$) are optimal solutions for the problem. The inner points ($\alpha < 1$) will have two zero queue length periods, one zero queue period for each movement.

In the case when the feasibility condition (Eq. 30) is satisfied by the equality constraint, i.e. $\frac{\alpha_1(\tau_1)}{-\alpha_1(\tau_0)} = \frac{-\alpha_2(\tau_1)}{\alpha_2(\tau_0)}$, the two points A and B will be identical points which can be calculated by (44) or (45). In this case the optimal solution will not have any movement with zero queue length period. Based on the above explanation the following proposition is true.

Proposition 5: There is always an optimal solution with at most one zero queue length period.

According to the Proposition 3, the queue lengths $q_1(\tau_1) = 0$ and $q_2(\tau_2) = 0$ in the optimal cyclic solutions. We define the "optimal initial cyclic queue lengths" set as the calculated set of initial queue lengths by the optimal cyclic solutions, i.e. $q_1^{\mathrm{opt}}(\tau_0) = T_1\alpha_1(\tau_0)$ and $q_2^{\mathrm{opt}}(\tau_0) = 0$. When the initial queue lengths are not in the set of the optimal initial cyclic queue lengths, the N-stages control can be used in order to bring the given initial queue lengths after N-cycles to final queue lengths that are in the set of the optimal initial cyclic queue lengths.

V. N-stages Control

In the N-stages control problem we consider a finite number of switchings in the optimization procedure. Now we specifically consider the following problem: for a given integer N and a given starting time t_0 we want to compute an optimal switching sequence consisting of N cycles[4].
For the simplified isolated controlled intersection we formulate the problem for the following two cases.

A. The criterion is a strictly increasing function of the queue lengths

We use the DMP problem (8) - (12) to solve the optimal problem for N-stages control when the criterion J is a strictly increasing function of the queue lengths. In this case, each max equation can be relaxed to two inequality equations, which leads to the "Relaxed" Discrete-event Max-Plus (R-DMP) problem

$$\min_{\substack{T_1(0),\,T_2(0),\\ T_1(1),\,T_2(1),\,\cdots,\\ T_1(N-1),\,T_2(N-1)}} J \tag{46}$$

subject to

$$q_1(t_{2k+1}) \geq q_1(t_{2k}) + \alpha_1(t_{2k})\cdot T_1(k) \tag{47}$$

$$q_1(t_{2k+1}) \geq 0 \tag{48}$$

$$q_2(t_{2k+2}) \geq q_2(t_{2k+1}) + \alpha_2(t_{2k+1})\cdot T_2(k) \tag{49}$$

$$q_2(t_{2k+2}) \geq 0 \tag{50}$$

$$\text{and } (10), (11) \tag{51}$$

for $k = 0, 1, 2, \ldots, N-1$.

Proposition 6: If the criterion J is a strictly increasing function of the queue lengths, then any optimal solution of the R-DMP problem is also an optimal solution of the DMP problem.

[4]In the N-stages control, the queue length vector q is defined as: $(q_1(t_1), q_2(t_1), q_1(t_2), q_2(t_2), \cdots, q_1(t_{2k+1}), q_2(t_{2k+1}), q_1(t_{2k+2}), q_2(t_{2k+2}))$.

Proof: See the proof of Proposition 3.3 of [4] which also applies here. ■

So the R-DMP problem can be solved by linear programming when the criterion J is a strictly increasing linear function of the queue lengths.

1) Feasibility condition for N-stages control: In the following we show the feasibility condition for the N-stages control problem. Considering the following problem: for a given initial queue lengths for movements m_1 and m_2, we want to decrease the queue lengths after N-cycles. This is done by considering only the first cycle in the N-cycles sequence. Given the initial queue lengths $q_1(t_0)$ and $q_2(t_0)$, we derive the conditions that should be satisfied in order to decrease the initial queue lengths by ε_1 and ε_2 for movements m_1 and m_2, respectively, i.e. $q_1(t_2) = q_1(t_0) - \varepsilon_1$ and $q_2(t_2) = q_2(t_0) - \varepsilon_2$ where $\varepsilon_1, \varepsilon_2 > 0$.
The following problem is given:

$$\min_{T_1(0),T_2(0)} J \tag{52}$$

subject to

$$q_1(t_1) \geq q_1(t_0) + \alpha_1(t_0) \cdot T_1(0) \tag{53}$$
$$q_1(t_1) \geq 0 \tag{54}$$
$$q_2(t_2) \geq q_2(t_1) + \alpha_2(t_1) \cdot T_2(0) \tag{55}$$
$$q_2(t_2) \geq 0 \tag{56}$$
$$q_1(t_2) = q_1(t_1) + \alpha_1(t_1) \cdot T_2(0) \tag{57}$$
$$q_2(t_1) = q_2(t_0) + \alpha_2(t_0) \cdot T_1(0) \tag{58}$$

For simplicity we denote $T_1(0)$ and $T_2(0)$ by T_1 and T_2, respectively. From these equation we get,

$$\alpha_1(t_1) \cdot T_2 + \alpha_1(t_0) \cdot T_1 \leq -\varepsilon_1 \tag{59}$$
$$\alpha_2(t_0) \cdot T_1 + \alpha_2(t_1) \cdot T_2 \leq -\varepsilon_2 \tag{60}$$

hence the feasibility condition is the following: Given an initial queue lengths, in order to get the optimal initial cyclic queue lengths, the following feasibility condition should be satisfied :

$$\frac{\alpha_1(\tau_1)}{-\alpha_1(\tau_0)} < \frac{-\alpha_2(\tau_1)}{\alpha_2(\tau_0)} \tag{61}$$

or by symmetry

$$\frac{-\alpha_1(\tau_0)}{\alpha_1(\tau_1)} > \frac{\alpha_2(\tau_0)}{-\alpha_2(\tau_1)} \tag{62}$$

it is the strictly case of the cyclic feasibility condition.

VI. Congestion on Isolated Controlled Intersections

The "congestion" on isolated controlled intersections is defined as the situation when the queue lengths at the intersection are increasing with time.
The classic congestion occurs when the arrival rate of one of the movements in the intersection is larger than the departure rate in the green light period, i.e. when Assumption A2 does not hold. Another case where congestion will occur even if the Assumption A2 holds, is when the feasibility condition for the N-stages (61) or (62) is violated.

VII. Conclusions and Future Research

For the simplified isolated controlled intersection we can compute the optimal switching sequences for the steady-state and N-stages control problems by solving a linear programming problem, a mixed-integer programming problem, or a mixed-integer linear programming problem.

A feasibility condition for the steady-state control has been derived when the criterion J is a strictly increasing function of the queue lengths. It is shown that if in addition the criterion is linear the problem can be solved analytically.

The N-stages control problem can be solved by linear programming if the criterion J is linear and strictly increasing.

VIII. Acknowledgments

This research was supported by the Technion - Israel Institute of Technology and the Henry Ford II transportation research fund.

References

[1] R. Allsop, "SIGSET: A computer program for calculating traffic capacity of signal-controlled road junctions," *Traffic Eng. Control*, vol. 12, pp. 58–60, 1971.
[2] ——, "SIGCAP: A computer program for assessing the traffic capacity of signal-controlled road junctions," *Traffic Eng. Control*, vol. 17, pp. 338–341, 1976.
[3] B. De Schutter, "Optimal control of a class of linear hybrid systems with saturation," *SIAM Journal on Control and Optimization*, vol. 39, pp. 835–851, 2000.
[4] ——, "Optimizing acyclic traffic signal switching sequences through an extended linear complementarity problem formulation," *European Journal of Operational Research*, vol. 139, pp. 400–415, 2002.
[5] B. De Schutter and B. De Moor, "Optimal traffic light control for a single intersection," *European Journal of Control*, vol. 4, no. 3, pp. 260–276, 1998.
[6] A. Di Febbraro, D. Giglio, and N. Sacco, "Urban traffic control structure based on hybrid Petri nets," *IEEE Transactions on Intelligent Transportation Systems*, vol. 5, no. 4, pp. 224–237, 2004.
[7] A. Di Febbraro and N. Sacco, "On modelling urban transportation networks via hybrid Petri nets," *Control Engineering Practice*, vol. 12, pp. 1225–1239, 2004.
[8] C. Diakaki, M. Papageorgiou, and K. Aboudolas, "A multivariable regulator approach to traffic-responsive network-wide signal control," *Control Engineering Practice*, vol. 10, pp. 183–195, 2002.
[9] M. Dotoli and M. Pia Fanti, "An urban traffic network model via coloured timed Petri nets," *Control Engineering Practice*, vol. 14, pp. 1213–1229, 2006.
[10] N. Gartner, J. Little, and H. Gabby, "Simultaneous optimization of offsets, splits and cycle time," *Transportation Research Record*, vol. 596, pp. 6–15, 1976.
[11] D. Gazis, "Optimum control of a system of oversaturated intersections," *Oper. Res.*, vol. 12, pp. 815–831, 1964.
[12] D. Gazis and R. Potts, "The oversaturated intersection," in *Proc. 2nd Int. Symp. on the Theory of Traffic Flow*, London, 1963, pp. 221–237.
[13] J. Haddad, B. De Schutter, D. Mahalel, and P. O. Gutman, "Steady-state and n-stages control for isolated controlled intersections," in *Submitted to The American Control Conference 2009*, St. Louis, Missouri, USA, 2009.
[14] G. Improta and G. Cantarella, "Control systems design for an individual signalised junction," *Transp. Res. B*, vol. 18, pp. 147–167, 1984.
[15] H. Kashani and G. Saridis, "Intelligent control for urban traffic systems," *Automatica*, vol. 19, no. 2, pp. 191–197, 1983.
[16] J. Lei and U. Ozguner, "Decentralized hybrid intersection control," in *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, Florida USA, December 2001, pp. 1237–1242.
[17] J. Lim, S. Hwang, I. Suh, and Z. Bien, "Hierarchical optimal control of oversaturated urban traffic networks," *International Journal of Control*, vol. 33 (4), pp. 727–737, 1981.
[18] J. Little, "The synchronization of traffic signal by mixed-integer-linear programming," *Oper. Res.*, vol. 14, pp. 568–594, 1966.

[19] M. Papageorgiou, C. Diakaki, V. Dinopoulou, A. Kotsialos, and Y. Wang, "Review of road traffic control strategies," *Proceedings of the IEEE*, vol. 91, no. 12, December 2003.

[20] D. I. Roberston, "TRANSYT method for area traffic control," *Traffic Eng. Control*, vol. 10, pp. 276–281, 1969.

[21] D. Schrank and T. Lomax, "2007 urban mobility report and appendices," Texas Transportation Institute, The Texas A&M University System, http://mobility.tamu.edu, Tech. Rep., 2007.

[22] I. Talmor and D. Mahalel, "Signal design for an isolated intersection during congestion," *Journal of the Operational Research Society*, vol. 58, pp. 454–466, 2007.

[23] I. A. Villalobos, A. S. Poznyak, and A. M. Tamayo, "Urban traffic control problem: a game theory approach," in *Proceedings of the 17th World Congress, The International Federation of Automatic Control*, Seoul, Korea, July 2008.