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Lagrangian Equilibrium Equations in Cylindrical and Spherical Coordinates

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Abstract: Lagrangian or referential equilibrium equations for materials undergoing large deformations are of interest in the developing fields of mechanics of soft biomaterials and nanomechanics. The main feature of these equations is the necessity to deal with the First Piola-Kirchhoff, or nominal, stress tensor which is a two-point tensor referring simultaneously to the reference and current configurations. This two-point nature of the First Piola-Kirchhoff tensor is not always appreciated by the researchers and the *total* covariant derivative necessary for the formulation of the equilibrium equations in curvilinear coordinates is sometimes inaccurately confused with the regular covariant derivative. Surprisingly, the traditional continuum mechanics literature does not discuss this issue properly, except for some brief notions on the two-point nature of the Piola-Kirchhoff tensor. We aim at partially filling this gap by giving a full yet simple derivation of the Lagrangian equilibrium equations in cylindrical and spherical coordinates.

1 Introduction

Lagrangian scalar equilibrium equations in cylindrical and spherical coordinates for materials undergoing large deformations are rarely discussed in the literature. The most influential monographs on nonlinear elasticity and continuum mechanics, including Antman (1995); Chadwick (1976); Ciarlet (1988); Eringen (1962); Green and Adkins (1970); Green and Zerna (1968); Gurtin (1981); Haupt (2000); Jaunzemis (1967); Liu (2002); Lur'e (1990); Malvern (1969); Marsden and Hughes (1983); Ogden (1984); Truesdell and Toupin (1961); Truesdell and Noll (1965); Wang and Truesdell (1973); Wilmanski (1998), do not address this issue. However, the Lagrangian equilibrium equations in cylindrical and spherical coordinates can be very useful in solving nonlinear problems analytically or semi-analytically. Sometimes, it is possible to assume incompressibility of the material what allows for using a simpler Eulerian description for obtaining some elementary analytical solutions. This is not the general case, however, where we need the Lagrangian equilibrium equations of the form

$$\mathrm{Div}\mathbf{P} = \mathbf{0} \tag{1}$$

in cylindrical and spherical coordinates. These equations can be derived from the *total covariant derivative* of the 1^{st} Piola-Kirchhoff stress tensor $\mathbf{P}$. Though this way may be elegant we prefer a more straightforward "pedestrian" way, which, however, does not require any knowledge of the general tensor calculus from the reader.

2 Cylindrical coordinates

We introduce orthonormal basis in cylindrical coordinates (Malvern, 1969) for the reference configuration

$$\begin{aligned}\mathbf{K}_R &= (\cos\Theta, \sin\Theta, 0)^T;\\ \mathbf{K}_\Theta &= (-\sin\Theta, \cos\Theta, 0)^T;\\ \mathbf{K}_Z &= (0,0,1)^T.\end{aligned} \tag{2}$$

By direct calculation we have

$$\frac{\partial \mathbf{K}_R}{\partial \Theta} = \mathbf{K}_\Theta; \quad \frac{\partial \mathbf{K}_\Theta}{\partial \Theta} = -\mathbf{K}_R. \tag{3}$$

All other derivatives of the base vectors are equal to zero. Analogously, we have for the current configuration:

$$\begin{aligned}\mathbf{k}_r &= (\cos\theta, \sin\theta, 0)^T;\\ \mathbf{k}_\theta &= (-\sin\theta, \cos\theta, 0)^T;\\ \mathbf{k}_z &= (0,0,1)^T,\end{aligned} \tag{4}$$

$$\frac{\partial \mathbf{k}_r}{\partial \theta} = \mathbf{k}_\theta; \quad \frac{\partial \mathbf{k}_\theta}{\partial \theta} = -\mathbf{k}_r. \tag{5}$$

Now, we write the divergence operator in the form (Malvern, 1969)

$$\mathrm{Div}\mathbf{P} = \frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R + \frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta + \frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z. \tag{6}$$

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The plan is to compute the right-hand side of this equation term by term.

We start with the first term on the right hand side of Eq. (6)

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R
&= \left(\tfrac{\partial P_{rR}}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial R} \otimes \mathbf{K}_R + P_{rR}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{r\Theta}}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial R} \otimes \mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{rZ}}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_Z + P_{rZ}\tfrac{\partial \mathbf{k}_r}{\partial R} \otimes \mathbf{K}_Z + P_{rZ}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_Z}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta \otimes \mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial R} \otimes \mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{\theta\Theta}}{\partial R}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial R} \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{\theta Z}}{\partial R}\mathbf{k}_\theta \otimes \mathbf{K}_Z + P_{\theta Z}\tfrac{\partial \mathbf{k}_\theta}{\partial R} \otimes \mathbf{K}_Z + P_{\theta Z}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_Z}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{zR}}{\partial R}\mathbf{k}_z \otimes \mathbf{K}_R + P_{zR}\tfrac{\partial \mathbf{k}_z}{\partial R} \otimes \mathbf{K}_R + P_{zR}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{z\Theta}}{\partial R}\mathbf{k}_z \otimes \mathbf{K}_\Theta + P_{z\Theta}\tfrac{\partial \mathbf{k}_z}{\partial R} \otimes \mathbf{K}_\Theta + P_{z\Theta}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{zZ}}{\partial R}\mathbf{k}_z \otimes \mathbf{K}_Z + P_{zZ}\tfrac{\partial \mathbf{k}_z}{\partial R} \otimes \mathbf{K}_Z + P_{zZ}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_Z}{\partial R}\right)\mathbf{K}_R,
\end{aligned}
\tag{7}
$$

where $\mathbf{k}_m \otimes \mathbf{K}_N = \mathbf{k}_m\mathbf{K}_N^T$.

With account of orthonormality of the base vectors we have

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R &= \frac{\partial P_{rR}}{\partial R}\mathbf{k}_r + P_{rR}\frac{\partial \mathbf{k}_r}{\partial R}\\
&+ \frac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta + P_{\theta R}\frac{\partial \mathbf{k}_\theta}{\partial R} + \frac{\partial P_{zR}}{\partial R}\mathbf{k}_z + P_{zR}\frac{\partial \mathbf{k}_z}{\partial R}.
\end{aligned}
\tag{8}
$$

Differentiating the Eulerian basis, we get

$$
\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial R} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial z}\frac{\partial z}{\partial R} = \frac{\partial \theta}{\partial R}\mathbf{k}_\theta,\\
\frac{\partial \mathbf{k}_\theta}{\partial R} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial z}\frac{\partial z}{\partial R} = -\frac{\partial \theta}{\partial R}\mathbf{k}_r,\\
\frac{\partial \mathbf{k}_z}{\partial R} &= \frac{\partial \mathbf{k}_z}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_z}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_z}{\partial z}\frac{\partial z}{\partial R} = \mathbf{0}.
\end{aligned}
\tag{9}
$$

Now, substituting Eq. (9) in Eq. (8) we have

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R &= \left(\frac{\partial P_{rR}}{\partial R} - P_{\theta R}\frac{\partial \theta}{\partial R}\right)\mathbf{k}_r\\
&+ \left(P_{rR}\frac{\partial \theta}{\partial R} + \frac{\partial P_{\theta R}}{\partial R}\right)\mathbf{k}_\theta + \frac{\partial P_{zR}}{\partial R}\mathbf{k}_z.
\end{aligned}
\tag{10}
$$

Analogously to Eqs. (7)-(10) we calculate the last two terms on the right-hand side of Eq. (6)

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta
&= \tfrac{1}{R}\left(\tfrac{\partial P_{rR}}{\partial \Theta}\mathbf{k}_r \otimes \mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial \Theta} \otimes \mathbf{K}_R + P_{rR}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{r\Theta}}{\partial \Theta}\mathbf{k}_r \otimes \mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial \Theta} \otimes \mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{rZ}}{\partial \Theta}\mathbf{k}_r \otimes \mathbf{K}_Z + P_{rZ}\tfrac{\partial \mathbf{k}_r}{\partial \Theta} \otimes \mathbf{K}_Z + P_{rZ}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_Z}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta R}}{\partial \Theta}\mathbf{k}_\theta \otimes \mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta} \otimes \mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta\Theta}}{\partial \Theta}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta} \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta Z}}{\partial \Theta}\mathbf{k}_\theta \otimes \mathbf{K}_Z + P_{\theta Z}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta} \otimes \mathbf{K}_Z + P_{\theta Z}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_Z}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{zR}}{\partial \Theta}\mathbf{k}_z \otimes \mathbf{K}_R + P_{zR}\tfrac{\partial \mathbf{k}_z}{\partial \Theta} \otimes \mathbf{K}_R + P_{zR}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{z\Theta}}{\partial \Theta}\mathbf{k}_z \otimes \mathbf{K}_\Theta + P_{z\Theta}\tfrac{\partial \mathbf{k}_z}{\partial \Theta} \otimes \mathbf{K}_\Theta + P_{z\Theta}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{zZ}}{\partial \Theta}\mathbf{k}_z \otimes \mathbf{K}_Z + P_{zZ}\tfrac{\partial \mathbf{k}_z}{\partial \Theta} \otimes \mathbf{K}_Z + P_{zZ}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_Z}{\partial \Theta}\right)\mathbf{K}_\Theta,
\end{aligned}
\tag{11}
$$

$$
\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta = \frac{1}{R}\begin{pmatrix} P_{rR}\mathbf{k}_r + \frac{\partial P_{r\Theta}}{\partial \Theta}\mathbf{k}_r + P_{r\Theta}\frac{\partial \mathbf{k}_r}{\partial \Theta}\\ +P_{\theta R}\mathbf{k}_\theta + \frac{\partial P_{\theta\Theta}}{\partial \Theta}\mathbf{k}_\theta + P_{\theta\Theta}\frac{\partial \mathbf{k}_\theta}{\partial \Theta}\\ +P_{zR}\mathbf{k}_z + \frac{\partial P_{z\Theta}}{\partial \Theta}\mathbf{k}_z + P_{z\Theta}\frac{\partial \mathbf{k}_z}{\partial \Theta} \end{pmatrix}
\tag{12}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial \Theta} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial z}\frac{\partial z}{\partial \Theta} = \frac{\partial \theta}{\partial \Theta}\mathbf{k}_\theta,\\
\frac{\partial \mathbf{k}_\theta}{\partial \Theta} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial z}\frac{\partial z}{\partial \Theta} = -\frac{\partial \theta}{\partial \Theta}\mathbf{k}_r,\\
\frac{\partial \mathbf{k}_z}{\partial \Theta} &= \frac{\partial \mathbf{k}_z}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_z}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_z}{\partial z}\frac{\partial z}{\partial \Theta} = \mathbf{0},
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta &= \left(\frac{P_{rR}}{R} + \frac{\partial P_{r\Theta}}{R\partial \Theta} - \frac{P_{\theta\Theta}}{R}\frac{\partial \theta}{\partial \Theta}\right)\mathbf{k}_r\\
&+ \left(\frac{P_{r\Theta}}{R}\frac{\partial \theta}{\partial \Theta} + \frac{P_{\theta R}}{R} + \frac{\partial P_{\theta\Theta}}{R\partial \Theta}\right)\mathbf{k}_\theta + \left(\frac{P_{zR}}{R} + \frac{\partial P_{z\Theta}}{R\partial \Theta}\right)\mathbf{k}_z
\end{aligned}
\tag{14}
$$

$$\frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z$$
$$= \left(\tfrac{\partial P_{rR}}{\partial Z}\mathbf{k}_r \otimes \mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial Z} \otimes \mathbf{K}_R + P_{rR}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_R}{\partial Z}\right)\mathbf{K}_Z$$
$$+ \left(\tfrac{\partial P_{r\Theta}}{\partial Z}\mathbf{k}_r \otimes \mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial Z} \otimes \mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial Z}\right)\mathbf{K}_Z$$
$$+ \left(\tfrac{\partial P_{rZ}}{\partial Z}\mathbf{k}_r \otimes \mathbf{K}_Z + P_{rZ}\tfrac{\partial \mathbf{k}_r}{\partial Z} \otimes \mathbf{K}_Z + P_{rZ}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_Z}{\partial Z}\right)\mathbf{K}_Z$$
$$+ \left(\tfrac{\partial P_{\theta R}}{\partial Z}\mathbf{k}_\theta \otimes \mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial Z} \otimes \mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_R}{\partial Z}\right)\mathbf{K}_Z$$
$$+ \left(\tfrac{\partial P_{\theta\Theta}}{\partial Z}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial Z} \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial Z}\right)\mathbf{K}_Z$$
$$+ \left(\tfrac{\partial P_{\theta Z}}{\partial Z}\mathbf{k}_\theta \otimes \mathbf{K}_Z + P_{\theta Z}\tfrac{\partial \mathbf{k}_\theta}{\partial Z} \otimes \mathbf{K}_Z + P_{\theta Z}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_Z}{\partial Z}\right)\mathbf{K}_Z$$
$$+ \left(\tfrac{\partial P_{zR}}{\partial Z}\mathbf{k}_z \otimes \mathbf{K}_R + P_{zR}\tfrac{\partial \mathbf{k}_z}{\partial Z} \otimes \mathbf{K}_R + P_{zR}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_R}{\partial Z}\right)\mathbf{K}_Z$$
$$+ \left(\tfrac{\partial P_{z\Theta}}{\partial Z}\mathbf{k}_z \otimes \mathbf{K}_\Theta + P_{z\Theta}\tfrac{\partial \mathbf{k}_z}{\partial Z} \otimes \mathbf{K}_\Theta + P_{z\Theta}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial Z}\right)\mathbf{K}_Z$$
$$+ \left(\tfrac{\partial P_{zZ}}{\partial Z}\mathbf{k}_z \otimes \mathbf{K}_Z + P_{zZ}\tfrac{\partial \mathbf{k}_z}{\partial Z} \otimes \mathbf{K}_Z + P_{zZ}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_Z}{\partial Z}\right)\mathbf{K}_Z, \tag{15}$$

$$\frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z = \frac{\partial P_{rZ}}{\partial Z}\mathbf{k}_r + P_{rZ}\frac{\partial \mathbf{k}_r}{\partial Z} + \frac{\partial P_{\theta Z}}{\partial Z}\mathbf{k}_\theta + P_{\theta Z}\frac{\partial \mathbf{k}_\theta}{\partial Z} + \frac{\partial P_{zZ}}{\partial Z}\mathbf{k}_z + P_{zZ}\frac{\partial \mathbf{k}_z}{\partial Z}, \tag{16}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial Z} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial Z} + \frac{\partial \mathbf{k}_r}{\partial z}\frac{\partial z}{\partial Z} = \frac{\partial \theta}{\partial Z}\mathbf{k}_\theta,\\
\frac{\partial \mathbf{k}_\theta}{\partial Z} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial Z} + \frac{\partial \mathbf{k}_\theta}{\partial z}\frac{\partial z}{\partial Z} = -\frac{\partial \theta}{\partial Z}\mathbf{k}_r,\\
\frac{\partial \mathbf{k}_z}{\partial Z} &= \frac{\partial \mathbf{k}_z}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial \mathbf{k}_z}{\partial \theta}\frac{\partial \theta}{\partial Z} + \frac{\partial \mathbf{k}_z}{\partial z}\frac{\partial z}{\partial Z} = \mathbf{0},
\end{aligned} \tag{17}$$

$$\frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z = \left(\frac{\partial P_{rZ}}{\partial Z} - P_{\theta Z}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_r + \left(\frac{\partial P_{\theta Z}}{\partial Z} + P_{rZ}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_\theta + \frac{\partial P_{zZ}}{\partial Z}\mathbf{k}_z. \tag{18}$$

Finally, substituting Eqs. (10), (14), and (18) in Eq. (6) we have

$$\begin{aligned}
\mathrm{Div}\mathbf{P} = &\left(\frac{\partial P_{rR}}{\partial R} - P_{\theta R}\frac{\partial \theta}{\partial R} + \frac{P_{rR}}{R} + \frac{\partial P_{r\Theta}}{R\partial \Theta} - \frac{P_{\theta\Theta}}{R}\frac{\partial \theta}{\partial \Theta} + \frac{\partial P_{rZ}}{\partial Z} - P_{\theta Z}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_r\\
&+ \left(P_{rR}\frac{\partial \theta}{\partial R} + \frac{\partial P_{\theta R}}{\partial R} + \frac{P_{r\Theta}}{R}\frac{\partial \theta}{\partial \Theta} + \frac{P_{\theta R}}{R} + \frac{\partial P_{\theta\Theta}}{R\partial \Theta} + \frac{\partial P_{\theta Z}}{\partial Z} + P_{rZ}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_\theta +\\
&\left(\frac{\partial P_{zR}}{\partial R} + \frac{P_{zR}}{R} + \frac{\partial P_{z\Theta}}{R\partial \Theta} + \frac{\partial P_{zZ}}{\partial Z}\right)\mathbf{k}_z
\end{aligned} \tag{19}$$

3 Spherical coordinates

We introduce orthonormal basis in spherical coordinates (Malvern, 1969) for the reference configuration

$$\begin{aligned}
\mathbf{K}_R &= (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)^T\\
\mathbf{K}_\Theta &= (\cos\Theta\cos\Phi, \cos\Theta\sin\Phi, -\sin\Theta)^T\\
\mathbf{K}_\Phi &= (-\sin\Phi, \cos\Phi, 0)^T.
\end{aligned} \tag{20}$$

By direct calculation we have the following nonzero derivatives of the base vectors

$$\begin{aligned}
&\frac{\partial \mathbf{K}_R}{\partial \Theta} = \mathbf{K}_\Theta; \quad \frac{\partial \mathbf{K}_\Theta}{\partial \Theta} = -\mathbf{K}_R; \quad \frac{\partial \mathbf{K}_R}{\partial \Phi} = \sin\Theta\mathbf{K}_\Phi;\\
&\frac{\partial \mathbf{K}_\Theta}{\partial \Phi} = \cos\Theta\mathbf{K}_\Phi; \quad \frac{\partial \mathbf{K}_\Phi}{\partial \Phi} = -\sin\Theta\mathbf{K}_R - \cos\Theta\mathbf{K}_\Theta.
\end{aligned} \tag{21}$$

Analogously, we have for the current configuration:

$$\begin{aligned}
\mathbf{k}_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T\\
\mathbf{k}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)^T\\
\mathbf{k}_\phi &= (-\sin\phi, \cos\phi, 0)^T,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\frac{\partial \mathbf{k}_r}{\partial \theta} = \mathbf{k}_\theta; \quad \frac{\partial \mathbf{k}_\theta}{\partial \theta} = -\mathbf{k}_r; \quad \frac{\partial \mathbf{k}_r}{\partial \phi} = \sin\theta\mathbf{k}_\phi;\\
&\frac{\partial \mathbf{k}_\theta}{\partial \phi} = \cos\theta\mathbf{k}_\phi; \quad \frac{\partial \mathbf{k}_\phi}{\partial \phi} = -\sin\theta\mathbf{k}_r - \cos\theta\mathbf{k}_\theta.
\end{aligned} \tag{23}$$

We will use the following abbreviation for the sake of simplicity

$$S \equiv \sin\Theta; \quad C \equiv \cos\Theta; \quad s \equiv \sin\theta; \quad c \equiv \cos\theta. \tag{24}$$

Now, we write the divergence operator in the form (Malvern, 1969)

$$\mathrm{Div}\mathbf{P} = \frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R + \frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta + \frac{\partial \mathbf{P}}{RS\partial \Phi}\mathbf{K}_\Phi. \tag{25}$$

The plan is again to compute the right-hand side of this equation term by term.

We start with

$$\begin{aligned}
&\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R \\
&= \left(\tfrac{\partial P_{rR}}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial R} \otimes \mathbf{K}_R + P_{rR}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{r\Theta}}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial R} \otimes \mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{r\Phi}}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_\Phi + P_{r\Phi}\tfrac{\partial \mathbf{k}_r}{\partial R} \otimes \mathbf{K}_\Phi + P_{r\Phi}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta \otimes \mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial R} \otimes \mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\theta\Theta}}{\partial R}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial R} \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\theta\Phi}}{\partial R}\mathbf{k}_\theta \otimes \mathbf{K}_\Phi + P_{\theta\Phi}\tfrac{\partial \mathbf{k}_\theta}{\partial R} \otimes \mathbf{K}_\Phi + P_{\theta\Phi}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\phi R}}{\partial R}\mathbf{k}_\phi \otimes \mathbf{K}_R + P_{\phi R}\tfrac{\partial \mathbf{k}_\phi}{\partial R} \otimes \mathbf{K}_R + P_{\phi R}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\phi\Theta}}{\partial R}\mathbf{k}_\phi \otimes \mathbf{K}_\Theta + P_{\phi\Theta}\tfrac{\partial \mathbf{k}_\phi}{\partial R} \otimes \mathbf{K}_\Theta + P_{\phi\Theta}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\phi\Phi}}{\partial R}\mathbf{k}_\phi \otimes \mathbf{K}_\Phi + P_{\phi\Phi}\tfrac{\partial \mathbf{k}_\phi}{\partial R} \otimes \mathbf{K}_\Phi + P_{\phi\Phi}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial R}\right)\mathbf{K}_R \\
&.
\end{aligned} \tag{26}$$

With account of orthonormality of the base vectors we have

$$\begin{aligned}
\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R &= \frac{\partial P_{rR}}{\partial R}\mathbf{k}_r + P_{rR}\frac{\partial \mathbf{k}_r}{\partial R} + \frac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta \\
&+ P_{\theta R}\frac{\partial \mathbf{k}_\theta}{\partial R} + \frac{\partial P_{\phi R}}{\partial R}\mathbf{k}_\phi + P_{\phi R}\frac{\partial \mathbf{k}_\phi}{\partial R}.
\end{aligned} \tag{27}$$

Differentiating the Eulerian basis, we get

$$\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial R} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial \phi}\frac{\partial \phi}{\partial R} = \frac{\partial \theta}{\partial R}\mathbf{k}_\theta + s\frac{\partial \phi}{\partial R}\mathbf{k}_\phi \\
\frac{\partial \mathbf{k}_\theta}{\partial R} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial \phi}\frac{\partial \phi}{\partial R} = -\frac{\partial \theta}{\partial R}\mathbf{k}_r + c\frac{\partial \phi}{\partial R}\mathbf{k}_\phi, \\
\frac{\partial \mathbf{k}_\phi}{\partial R} &= \frac{\partial \mathbf{k}_\phi}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_\phi}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_\phi}{\partial \phi}\frac{\partial \phi}{\partial R} = -s\frac{\partial \phi}{\partial R}\mathbf{k}_r - c\frac{\partial \phi}{\partial R}\mathbf{k}_\theta.
\end{aligned} \tag{28}$$

Now, substituting Eq. (28) in Eq. (27) we have

$$\begin{aligned}
\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R &= \left(\frac{\partial P_{rR}}{\partial R} - P_{\theta R}\frac{\partial \theta}{\partial R} - sP_{\phi R}\frac{\partial \phi}{\partial R}\right)\mathbf{k}_r \\
&+ \left(\frac{\partial P_{\theta R}}{\partial R} + P_{rR}\frac{\partial \theta}{\partial R} - cP_{\phi R}\frac{\partial \phi}{\partial R}\right)\mathbf{k}_\theta \\
&+ \left(\frac{\partial P_{\phi R}}{\partial R} + sP_{rR}\frac{\partial \phi}{\partial R} + cP_{\theta R}\frac{\partial \phi}{\partial R}\right)\mathbf{k}_\phi.
\end{aligned} \tag{29}$$

Analogously to Eqs. (26)-(29) we calculate the last two terms on the right-hand side of Eq. (25)

$$\begin{aligned}
&\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta \\
&= \tfrac{1}{R}\left(\tfrac{\partial P_{rR}}{\partial \Theta}\mathbf{k}_r \otimes \mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial \Theta} \otimes \mathbf{K}_R + P_{rR}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{r\Theta}}{\partial \Theta}\mathbf{k}_r \otimes \mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial \Theta} \otimes \mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{r\Phi}}{\partial \Theta}\mathbf{k}_r \otimes \mathbf{K}_\Phi + P_{r\Phi}\tfrac{\partial \mathbf{k}_r}{\partial \Theta} \otimes \mathbf{K}_\Phi + P_{r\Phi}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta R}}{\partial \Theta}\mathbf{k}_\theta \otimes \mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta} \otimes \mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta\Theta}}{\partial \Theta}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta} \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta\Phi}}{\partial \Theta}\mathbf{k}_\theta \otimes \mathbf{K}_\Phi + P_{\theta\Phi}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta} \otimes \mathbf{K}_\Phi + P_{\theta\Phi}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\phi R}}{\partial \Theta}\mathbf{k}_\phi \otimes \mathbf{K}_R + P_{\phi R}\tfrac{\partial \mathbf{k}_\phi}{\partial \Theta} \otimes \mathbf{K}_R + P_{\phi R}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\phi\Theta}}{\partial \Theta}\mathbf{k}_\phi \otimes \mathbf{K}_\Theta + P_{\phi\Theta}\tfrac{\partial \mathbf{k}_\phi}{\partial \Theta} \otimes \mathbf{K}_\Theta + P_{\phi\Theta}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\phi\Phi}}{\partial \Theta}\mathbf{k}_\phi \otimes \mathbf{K}_\Phi + P_{\phi\Phi}\tfrac{\partial \mathbf{k}_\phi}{\partial \Theta} \otimes \mathbf{K}_\Phi + P_{\phi\Phi}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial \Theta}\right)\mathbf{K}_\Theta
\end{aligned} \tag{30}$$

$$\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta = \frac{1}{R}\left(\begin{array}{l}
P_{rR}\mathbf{k}_r + \frac{\partial P_{r\Theta}}{\partial \Theta}\mathbf{k}_r + P_{r\Theta}\frac{\partial \mathbf{k}_r}{\partial \Theta} + P_{\theta R}\mathbf{k}_\theta \\
+\frac{\partial P_{\theta\Theta}}{\partial \Theta}\mathbf{k}_\theta + P_{\theta\Theta}\frac{\partial \mathbf{k}_\theta}{\partial \Theta} + P_{\phi R}\mathbf{k}_\phi \\
+\frac{\partial P_{\phi\Theta}}{\partial \Theta}\mathbf{k}_\phi + P_{\phi\Theta}\frac{\partial \mathbf{k}_\phi}{\partial \Theta}
\end{array}\right), \tag{31}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial \Theta} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial \phi}\frac{\partial \phi}{\partial \Theta} = \frac{\partial \theta}{\partial \Theta}\mathbf{k}_\theta + s\frac{\partial \phi}{\partial \Theta}\mathbf{k}_\phi, \\
\frac{\partial \mathbf{k}_\theta}{\partial \Theta} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial \phi}\frac{\partial \phi}{\partial \Theta} = -\frac{\partial \theta}{\partial \Theta}\mathbf{k}_r + c\frac{\partial \phi}{\partial \Theta}\mathbf{k}_\phi, \\
\frac{\partial \mathbf{k}_\phi}{\partial \Theta} &= \frac{\partial \mathbf{k}_\phi}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_\phi}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_\phi}{\partial \phi}\frac{\partial \phi}{\partial \Theta} = -s\frac{\partial \phi}{\partial \Theta}\mathbf{k}_r - c\frac{\partial \phi}{\partial \Theta}\mathbf{k}_\theta,
\end{aligned} \tag{32}$$

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{R\partial\Theta}\mathbf{K}_\Theta = &\left(\frac{P_{rR}}{R}+\frac{\partial P_{r\Theta}}{R\partial\Theta}-\frac{P_{\theta\Theta}}{R}\frac{\partial\theta}{\partial\Theta}-s\frac{P_{\phi\Theta}}{R}\frac{\partial\phi}{\partial\Theta}\right)\mathbf{k}_r \\
&+\left(\frac{P_{\theta R}}{R}+\frac{\partial P_{\theta\Theta}}{R\partial\Theta}+\frac{P_{r\Theta}}{R}\frac{\partial\theta}{\partial\Theta}-c\frac{P_{\phi\Theta}}{R}\frac{\partial\phi}{\partial\Theta}\right)\mathbf{k}_\theta \\
&+\left(\frac{P_{\phi R}}{R}+\frac{\partial P_{\phi\Theta}}{R\partial\Theta}+s\frac{P_{r\Theta}}{R}\frac{\partial\phi}{\partial\Theta}+c\frac{P_{\theta\Theta}}{R}\frac{\partial\phi}{\partial\Theta}\right)\mathbf{k}_\phi,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&\frac{\partial \mathbf{P}}{RS\partial\Phi}\mathbf{K}_\Phi \\
&= \tfrac{1}{RS}\left(\tfrac{\partial P_{rR}}{\partial\Phi}\mathbf{k}_r\otimes\mathbf{K}_R+P_{rR}\tfrac{\partial\mathbf{k}_r}{\partial\Phi}\otimes\mathbf{K}_R+P_{rR}\mathbf{k}_r\otimes\tfrac{\partial\mathbf{K}_R}{\partial\Phi}\right)\mathbf{K}_\Phi \\
&+\tfrac{1}{RS}\left(\tfrac{\partial P_{r\Theta}}{\partial\Phi}\mathbf{k}_r\otimes\mathbf{K}_\Theta+P_{r\Theta}\tfrac{\partial\mathbf{k}_r}{\partial\Phi}\otimes\mathbf{K}_\Theta+P_{r\Theta}\mathbf{k}_r\otimes\tfrac{\partial\mathbf{K}_\Theta}{\partial\Phi}\right)\mathbf{K}_\Phi \\
&+\tfrac{1}{RS}\left(\tfrac{\partial P_{r\Phi}}{\partial\Phi}\mathbf{k}_r\otimes\mathbf{K}_\Phi+P_{r\Phi}\tfrac{\partial\mathbf{k}_r}{\partial\Phi}\otimes\mathbf{K}_\Phi+P_{r\Phi}\mathbf{k}_r\otimes\tfrac{\partial\mathbf{K}_\Phi}{\partial\Phi}\right)\mathbf{K}_\Phi \\
&+\tfrac{1}{RS}\left(\tfrac{\partial P_{\theta R}}{\partial\Phi}\mathbf{k}_\theta\otimes\mathbf{K}_R+P_{\theta R}\tfrac{\partial\mathbf{k}_\theta}{\partial\Phi}\otimes\mathbf{K}_R+P_{\theta R}\mathbf{k}_\theta\otimes\tfrac{\partial\mathbf{K}_R}{\partial\Phi}\right)\mathbf{K}_\Phi \\
&+\tfrac{1}{RS}\left(\tfrac{\partial P_{\theta\Theta}}{\partial\Phi}\mathbf{k}_\theta\otimes\mathbf{K}_\Theta+P_{\theta\Theta}\tfrac{\partial\mathbf{k}_\theta}{\partial\Phi}\otimes\mathbf{K}_\Theta+P_{\theta\Theta}\mathbf{k}_\theta\otimes\tfrac{\partial\mathbf{K}_\Theta}{\partial\Phi}\right)\mathbf{K}_\Phi \\
&+\tfrac{1}{RS}\left(\tfrac{\partial P_{\theta\Phi}}{\partial\Phi}\mathbf{k}_\theta\otimes\mathbf{K}_\Phi+P_{\theta\Phi}\tfrac{\partial\mathbf{k}_\theta}{\partial\Phi}\otimes\mathbf{K}_\Phi+P_{\theta\Phi}\mathbf{k}_\theta\otimes\tfrac{\partial\mathbf{K}_\Phi}{\partial\Phi}\right)\mathbf{K}_\Phi \\
&+\tfrac{1}{RS}\left(\tfrac{\partial P_{\phi R}}{\partial\Phi}\mathbf{k}_\phi\otimes\mathbf{K}_R+P_{\phi R}\tfrac{\partial\mathbf{k}_\phi}{\partial\Phi}\otimes\mathbf{K}_R+P_{\phi R}\mathbf{k}_\phi\otimes\tfrac{\partial\mathbf{K}_R}{\partial\Phi}\right)\mathbf{K}_\Phi \\
&+\tfrac{1}{RS}\left(\tfrac{\partial P_{\phi\Theta}}{\partial\Phi}\mathbf{k}_\phi\otimes\mathbf{K}_\Theta+P_{\phi\Theta}\tfrac{\partial\mathbf{k}_\phi}{\partial\Phi}\otimes\mathbf{K}_\Theta+P_{\phi\Theta}\mathbf{k}_\phi\otimes\tfrac{\partial\mathbf{K}_\Theta}{\partial\Phi}\right)\mathbf{K}_\Phi \\
&+\tfrac{1}{RS}\left(\tfrac{\partial P_{\phi\Phi}}{\partial\Phi}\mathbf{k}_\phi\otimes\mathbf{K}_\Phi+P_{\phi\Phi}\tfrac{\partial\mathbf{k}_\phi}{\partial\Phi}\otimes\mathbf{K}_\Phi+P_{\phi\Phi}\mathbf{k}_\phi\otimes\tfrac{\partial\mathbf{K}_\Phi}{\partial\Phi}\right)\mathbf{K}_\Phi,
\end{aligned}
\tag{34}
$$

$$
\frac{\partial \mathbf{P}}{RS\partial\Phi}\mathbf{K}_\Phi = \frac{1}{RS}\begin{pmatrix}
SP_{rR}\mathbf{k}_r+CP_{r\Theta}\mathbf{k}_r+\frac{\partial P_{r\Phi}}{\partial\Phi}\mathbf{k}_r \\
+P_{r\Phi}\frac{\partial\mathbf{k}_r}{\partial\Phi}+SP_{\theta R}\mathbf{k}_\theta+CP_{\theta\Theta}\mathbf{k}_\theta \\
+\frac{\partial P_{\theta\Phi}}{\partial\Phi}\mathbf{k}_\theta+P_{\theta\Phi}\frac{\partial\mathbf{k}_\theta}{\partial\Phi}+SP_{\phi R}\mathbf{k}_\phi \\
+CP_{\phi\Theta}\mathbf{k}_\phi+\frac{\partial P_{\phi\Phi}}{\partial\Phi}\mathbf{k}_\phi+P_{\phi\Phi}\frac{\partial\mathbf{k}_\phi}{\partial\Phi}
\end{pmatrix},
\tag{35}
$$

$$
\begin{aligned}
\frac{\partial\mathbf{k}_r}{\partial\Phi} &= \frac{\partial\mathbf{k}_r}{\partial r}\frac{\partial r}{\partial\Phi}+\frac{\partial\mathbf{k}_r}{\partial\theta}\frac{\partial\theta}{\partial\Phi}+\frac{\partial\mathbf{k}_r}{\partial\phi}\frac{\partial\phi}{\partial\Phi}=\frac{\partial\theta}{\partial\Phi}\mathbf{k}_\theta+s\frac{\partial\phi}{\partial\Phi}\mathbf{k}_\phi, \\
\frac{\partial\mathbf{k}_\theta}{\partial\Phi} &= \frac{\partial\mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial\Phi}+\frac{\partial\mathbf{k}_\theta}{\partial\theta}\frac{\partial\theta}{\partial\Phi}+\frac{\partial\mathbf{k}_\theta}{\partial\phi}\frac{\partial\phi}{\partial\Phi} \\
&= -\frac{\partial\theta}{\partial\Phi}\mathbf{k}_r+c\frac{\partial\phi}{\partial\Phi}\mathbf{k}_\phi, \\
\frac{\partial\mathbf{k}_\phi}{\partial\Phi} &= \frac{\partial\mathbf{k}_\phi}{\partial r}\frac{\partial r}{\partial\Phi}+\frac{\partial\mathbf{k}_\phi}{\partial\theta}\frac{\partial\theta}{\partial\Phi}+\frac{\partial\mathbf{k}_\phi}{\partial\phi}\frac{\partial\phi}{\partial\Phi} \\
&= -s\frac{\partial\phi}{\partial\Phi}\mathbf{k}_r-c\frac{\partial\phi}{\partial\Phi}\mathbf{k}_\theta,
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&\frac{\partial \mathbf{P}}{RS\partial\Phi}\mathbf{K}_\Phi \\
&= \left(\frac{P_{rR}}{R}+\frac{CP_{r\Theta}}{RS}+\frac{\partial P_{r\Phi}}{RS\partial\Phi}-\frac{P_{\theta\Phi}}{RS}\frac{\partial\theta}{\partial\Phi}-\frac{sP_{\phi\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}\right)\mathbf{k}_r \\
&+\left(\frac{P_{\theta R}}{R}+\frac{CP_{\theta\Theta}}{RS}+\frac{\partial P_{\theta\Phi}}{RS\partial\Phi}+\frac{P_{r\Phi}}{RS}\frac{\partial\theta}{\partial\Phi}-\frac{cP_{\phi\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}\right)\mathbf{k}_\theta \\
&+\left(\frac{P_{\phi R}}{R}+\frac{CP_{\phi\Theta}}{RS}+\frac{\partial P_{\phi\Phi}}{RS\partial\Phi}+\frac{sP_{r\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}+\frac{cP_{\theta\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}\right)\mathbf{k}_\phi.
\end{aligned}
\tag{37}
$$

Finally, substituting Eqs. (29), (33), and (37) in Eq. (25) we have

$$
\begin{aligned}
&\mathrm{Div}\mathbf{P} \\
&= \begin{pmatrix}
\frac{\partial P_{rR}}{\partial R}-P_{\theta R}\frac{\partial\theta}{\partial R}-sP_{\phi R}\frac{\partial\phi}{\partial R} \\
+2\frac{P_{rR}}{R}+\frac{\partial P_{r\Theta}}{R\partial\Theta}-\frac{P_{\theta\Theta}}{R}\frac{\partial\theta}{\partial\Theta} \\
-s\frac{P_{\phi\Theta}}{R}\frac{\partial\phi}{\partial\Theta}+\frac{CP_{r\Theta}}{RS}+\frac{\partial P_{r\Phi}}{RS\partial\Phi} \\
-\frac{P_{\theta\Phi}}{RS}\frac{\partial\theta}{\partial\Phi}-\frac{sP_{\phi\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}
\end{pmatrix}\mathbf{k}_r \\
&+ \begin{pmatrix}
\frac{\partial P_{\theta R}}{\partial R}+P_{rR}\frac{\partial\theta}{\partial R}-cP_{\phi R}\frac{\partial\phi}{\partial R}+ \\
2\frac{P_{\theta R}}{R}+\frac{\partial P_{\theta\Theta}}{R\partial\Theta}+\frac{P_{r\Theta}}{R}\frac{\partial\theta}{\partial\Theta} \\
-c\frac{P_{\phi\Theta}}{R}\frac{\partial\phi}{\partial\Theta}+\frac{CP_{\theta\Theta}}{RS}+\frac{\partial P_{\theta\Phi}}{RS\partial\Phi} \\
+\frac{P_{r\Phi}}{RS}\frac{\partial\theta}{\partial\Phi}-\frac{cP_{\phi\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}
\end{pmatrix}\mathbf{k}_\theta \\
&+ \begin{pmatrix}
\frac{\partial P_{\phi R}}{\partial R}+sP_{rR}\frac{\partial\phi}{\partial R}+cP_{\theta R}\frac{\partial\phi}{\partial R} \\
+2\frac{P_{\phi R}}{R}+\frac{\partial P_{\phi\Theta}}{R\partial\Theta}+s\frac{P_{r\Theta}}{R}\frac{\partial\phi}{\partial\Theta} \\
+c\frac{P_{\theta\Theta}}{R}\frac{\partial\phi}{\partial\Theta}+\frac{CP_{\phi\Theta}}{RS}+\frac{\partial P_{\phi\Phi}}{RS\partial\Phi} \\
+\frac{sP_{r\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}+\frac{cP_{\theta\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}
\end{pmatrix}\mathbf{k}_\phi.
\end{aligned}
\tag{38}
$$

4 Conclusion

Lagrangian equilibrium equations in cylindrical (Eq. 19) and spherical coordinates (Eq. 38) have been derived in the present work.

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