

An explanation of the drag reduction via polymer solute

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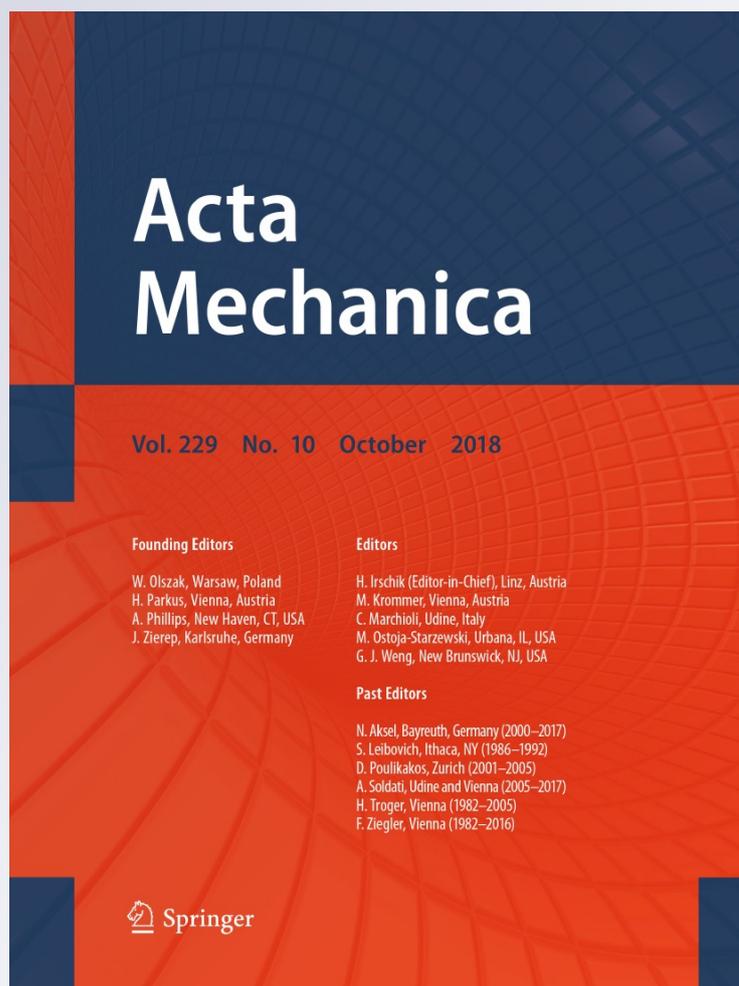
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**NOTE****K. Y. Volokh****An explanation of the drag reduction via polymer solute**Received: 16 April 2018 / Revised: 2 May 2018 / Published online: 27 July 2018
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Abstract The remarkable phenomenon of the drag reduction via addition of small amounts of polymer molecules to a Newtonian solvent was observed experimentally long ago. However, the theoretical explanations of this observation are not overwhelming yet. In this note, we present a possible theoretical account of the phenomenon. It is based on the use of the Navier–Stokes model with viscous strength for the solvent and the upper-convected Maxwell model for the polymer solute. Simple analytical calculation shows that the laminar flow of the solvent is stabilized by an addition of the polymer solute and, thus, the transition to the chaotic and slower on average turbulent motion is suppressed.

1 Introduction

Interesting things can happen to the flows of Newtonian fluids with addition of small amounts of various substances. It was Toms [23] who noticed, probably for the first time, that the addition of a small amount of a polymer solute could reduce drag in a pipe flow at high Reynolds numbers. In other words and contrary to intuition, Toms observed less resistance to the flow of the solution as compared to the flow of the pure solvent. Reducing the drag, it is possible to dramatically increase the rate of flow with exciting practical implications. It is not surprising, in view of the said, that there is an impressive body of scientific literature considering flows with additives. For example, the bibliography prepared by Nadolink and Haigh [15] counts some 2800 sources up to 1995. The knowledge concerning the drag reduction phenomenon was collected by Gyr and Bewersdorff [10] the same year. One can only imagine a further extrapolation of these numbers of references to the present. We mention a few works after 1995 by way of examples: [3, 4, 6, 9, 13, 17, 19, 20, 24, 29].

Despite well-reported observations of the drag reduction phenomenon, its theoretical explanation is still lacking. Of course, there is an objective difficulty in explaining the drag reduction phenomenon based on the theory of a turbulent flow—the latter theory has not been finalized yet. However, the understanding of the drag reduction on the conceptual level is also obscure. Probably, the first attempt to articulate a possible physical scenario of the phenomenon was made by Oldroyd [16] who suggested the flow slip at the wall under addition of the polymer solute. Tanner and Walters [22] note in this regard that “although it (the wall-slip hypothesis) may give a description of the drag reduction phenomenon, it is not a satisfactory explanation, since it is clear that the solvent does not slip at the wall”. A much more probable scenario of the drag reduction is the hypothesis that the additives suppress turbulent flow and friction, making the flow (almost) laminar and fast [14]. This hypothesis is rooted in experimental observations showing a significant decrease in the local chaotic turbulent motions in the presence of polymer molecules. We find this scenario of the suppression of turbulence appealing, and we suggest a simple theoretical explanation of it in the present note.

Since we are short of a final theory of the turbulent flow, it is reasonable to not try analyzing the reverse transition of the turbulent flow into the laminar one. Rather, it is rational to analyze the factors that can preclude the instabilities of laminar flow triggering turbulence. So, the instability of a Newtonian solvent should be considered in the presence of the polymer solute. The obstacle here is that the classical Navier–Stokes constitutive model of the solvent does not predict linear instabilities in the pipe flow. The latter is an obvious drawback of the classical theory because it contradicts experimental results with instabilities and transition to turbulence at Reynolds numbers around ~ 2000 . To overcome this drawback, it was assumed by some authors that finite initial perturbations could exist in the laminar flow in the form of coherent structures or traveling waves: [7, 8, 11, 27]. In the presence of finite perturbations of the laminar flow, the *nonlinear* instability analysis can be resorted to in order to explain transition to turbulence. The nonlinear instability analysis can be physically appealing under specific circumstances. Unfortunately, finite flow perturbations do not generally exist, while infinitesimal (molecular) perturbations always exist¹, and so one should expect linear instability analysis to catch the onset of the transition to turbulence.

A simple way to modify the classical Navier–Stokes theory is to relax the assumption of the *unbounded* linear dependence of stress on deformation rate. Volokh [25, 26] suggested to limit the linear stress–deformation-rate dependence by *viscous strength* beyond which the fluid would become ideal with negligible viscous friction. The constitutive theory with the bounded stress automatically allows for the local loss of material stability and transition to turbulence. The modified Navier–Stokes theory with the viscous strength is used in the present work for the solvent, while the standard upper-convected Maxwell theory is used for the polymer solute. The general constitutive equations are summarized in Sect. 2, and they are applied to the analysis of a pipe flow in Sect. 3. A brief discussion of the obtained results is presented in Sect. 4.

2 Constitutive model

We consider the following constitutive equation for the mixture of water and polymer molecules:

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\eta\mathbf{D} + \boldsymbol{\tau}, \quad (1)$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\boldsymbol{\tau}$ is the stress due to the polymer solute, p is the unknown pressure, $\mathbf{1}$ is the second-order identity tensor, η is the water viscosity, and \mathbf{D} is the deformation rate

$$2\mathbf{D} = \text{grad}\mathbf{v} + (\text{grad}\mathbf{v})^T. \quad (2)$$

The velocity field \mathbf{v} obeys the incompressibility condition

$$\text{div}\mathbf{v} = 0. \quad (3)$$

It is important to note that the water viscosity η is not constant. Rather, it is defined as follows [25, 26]:

$$\eta = \eta_0 \exp[-(\mathbf{D} : \mathbf{D} / \Phi^2)^m], \quad (4)$$

where $\mathbf{D} : \mathbf{D} = D_{ij}D_{ij}$ in Cartesian coordinates and the sum is performed over the repeated indices; $\eta_0 = 10^{-3} \text{ N s/m}^2$ is a constant (initial) water viscosity, and m and Φ are additional material constants.

It is seen from Fig. 1 that the viscosity coefficient has two main modes: linearly viscous fluid:

$$\eta = \eta_0 \quad \text{when} \quad \mathbf{D} : \mathbf{D} < \Phi^2, \quad (5)$$

and ideal fluid:

$$\eta = 0 \quad \text{when} \quad \mathbf{D} : \mathbf{D} > \Phi^2, \quad (6)$$

and by increasing m , it is possible to sharpen the step function.

The first mode corresponds to the classical Navier–Stokes model with viscous friction. The second mode corresponds to the loss of viscosity. These two modes reflect upon Landau's remark that "...for the large eddies which are the basis of any turbulent flow, the viscosity is unimportant" ([12], Section 33: "Fully developed turbulence").

¹ Molecules/atoms are never at rest. Even in solids, where they are well bonded, heat propagates due to the local molecular/atomic vibrations. In fluids, where molecules are much less bonded, it is natural to expect their constant local perturbed (infinitesimal) motion superimposed on the global flow.

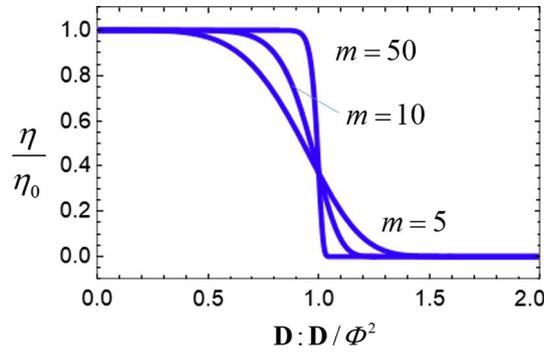


Fig. 1 The meaning of the viscosity coefficient

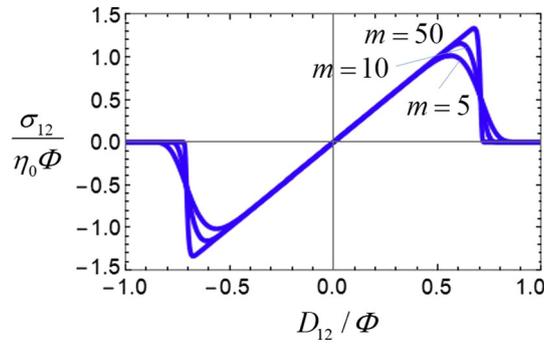


Fig. 2 Stress versus deformation rate in shear flow

For large m ($m \rightarrow \infty$), Volokh [26] calibrated $\Phi = 848.5\text{s}^{-1}$ for the critical Reynolds number $Re_{cr} \approx 2400$ based on the experiments by Avila et al. [1].

To show how the viscous strength works, it is instructive to consider the steady shear flow described by the deformation rate $\mathbf{D} = D_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$ and stress (without $\boldsymbol{\tau}$) $\boldsymbol{\sigma} = -p\mathbf{1} + \sigma_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$ tensors, where \mathbf{e}_i is a Cartesian basis vector. In this case, the constitutive law (1) takes the following reduced form: $\sigma_{12} = 2\eta_0 D_{12} \exp[-(2D_{12}^2/\Phi^2)^m]$; it is presented graphically in Fig. 2.

The limit points have the following horizontal coordinates: $D_{12}/\Phi = \pm 2^{-1/2}(2m)^{-1/(2m)}$. In the case of $m \rightarrow \infty$, we get the critical deformation rate $D_{12}^{cr} = \pm\Phi/\sqrt{2}$ and the critical stress $\sigma_{12}^{cr} = \pm\sqrt{2}\Phi\eta_0$.

It is interesting to note that the recent molecular dynamics simulations by Raghavan and Ostoja-Starzewski [21] favor the modified Navier–Stokes model with viscous strength.

It remains to define the constitutive law for the stress $\boldsymbol{\tau}$ due to the polymer solute—the upper-convected Maxwell model [5, 18]

$$\boldsymbol{\tau} + \lambda \overset{\diamond}{\boldsymbol{\tau}} = G\mathbf{1}, \tag{7}$$

where λ is the relaxation time; $G = n_p k_B T$ is the polymer shear modulus with n_p representing the number of molecular sub-chains (crosslinks) per unit volume, k_B designating the Boltzmann constant and T for the absolute temperature; and the Oldroyd objective rate of the stress is defined as follows:

$$\overset{\diamond}{\boldsymbol{\tau}} = \frac{\partial \boldsymbol{\tau}}{\partial t} + (\text{grad} \boldsymbol{\tau})\mathbf{v} - (\text{grad} \mathbf{v})\boldsymbol{\tau} - \boldsymbol{\tau}(\text{grad} \mathbf{v})^T. \tag{8}$$

Remark 1 We emphasize that the constitutive model for the solvent used in our work—the Navier–Stokes model with the viscous strength has clear experimental roots. For example, Fig. 2 of [2] shows the dependence of the friction factor on the Reynolds number in the pipe flow. The green colored points designate experiments. At the critical Reynolds numbers of about 2000, the laminar flow becomes unstable (white points) and the instability and transition to turbulence take place. Also, Wygnanski and Champagne [28] summarize the experimental data on the schematic diagram shown in Fig. 2(b) of their paper. Again, the pipe flow becomes unstable after passing the Reynolds number of approximately 2000. The instability appears in the form of slugs that are developed under very small disturbances. The classical Navier–Stokes model fails to describe

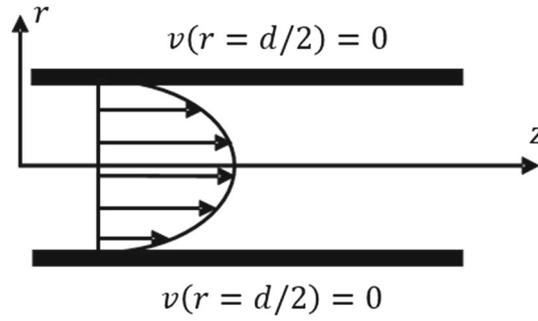


Fig. 3 Flow in a pipe

these instabilities. The reason is that the Navier–Stokes model can only capture *kinematic instabilities* of the flow. The generalized Navier–Stokes model with viscous strength bounds stresses, and consequently, it automatically introduces *material instabilities*, which trigger the transition to turbulence.

Remark 2 We should also emphasize that the traditional non-Newtonian fluid models with *shear thinning* are qualitatively different from the considered Navier–Stokes model with viscous strength. For example, in the case of shear flow, the limit points (Fig. 2) are absent in the traditional non-Newtonian or generalized Newtonian models. Shear thinning means the decrease in the positive slope of the stress–deformation rate curve rather than the transition of the slope from positive to negative. The latter transition produces the limit point, which is the indication of material instability. Contrary to the model considered in the present work, traditional shear thinning models do not introduce material instabilities.

3 Flow in a pipe

We consider the axisymmetric flow in a pipe—Fig. 3.

Neglecting body forces, the flow must obey the linear momentum balance equation

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\text{grad} \mathbf{v})\mathbf{v} = \text{div} \boldsymbol{\sigma}, \quad (9)$$

where $\rho \approx 10^3 \text{ kg/m}^3$ is the solution density which is approximately equal to the water density for small amounts of the added polymer.

We use cylindrical coordinates r, φ, z with the unit basis vectors

$$\mathbf{g}_r = \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2, \quad \mathbf{g}_\varphi = -\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2, \quad \mathbf{g}_z = \mathbf{e}_3. \quad (10)$$

We assume that the process is steady: $\partial \mathbf{v} / \partial t = \mathbf{0}$, and the distribution of velocities is axisymmetric,

$$\mathbf{v} = v(r)\mathbf{g}_z, \quad (11)$$

which obeys the incompressibility condition (3).

Under this assumption, the velocity gradient takes the form

$$\text{grad} \mathbf{v} = \frac{\partial v}{\partial r} \mathbf{g}_z \otimes \mathbf{g}_r, \quad (12)$$

and the deformation rate is

$$\mathbf{D} = \frac{1}{2} \frac{\partial v}{\partial r} (\mathbf{g}_z \otimes \mathbf{g}_r + \mathbf{g}_r \otimes \mathbf{g}_z). \quad (13)$$

With account of the kinematic assumptions, the momentum balance equation (9) becomes

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\varphi\varphi}) &= 0, \\ \frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{\partial \sigma_{\varphi z}}{\partial z} + \frac{2}{r}\sigma_{r\varphi} &= 0, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} &= 0. \end{aligned} \quad (14)$$

The constitutive equation (1) takes the componentwise form

$$\begin{aligned}
 \sigma_{rr} &= -p + \tau_{rr}, \\
 \sigma_{\varphi\varphi} &= -p + \tau_{\varphi\varphi}, \\
 \sigma_{zz} &= -p + \tau_{zz}, \\
 \sigma_{r\varphi} &= \tau_{r\varphi}, \\
 \sigma_{rz} &= \eta \frac{\partial v}{\partial r} + \tau_{rz}, \\
 \sigma_{\varphi z} &= \tau_{\varphi z}.
 \end{aligned} \tag{15}$$

We further assume that the components of tensor τ depend on the radial coordinate only and, consequently, its Oldroyd rate is calculated as follows:

$$\begin{aligned}
 \overset{\diamond}{\tau} &= -\frac{\partial v}{\partial r} \{(\mathbf{g}_z \otimes \mathbf{g}_r)\tau + \tau(\mathbf{g}_r \otimes \mathbf{g}_z)\} \\
 &= -\frac{\partial v}{\partial r} \{\tau_{rr}(\mathbf{g}_z \otimes \mathbf{g}_r + \mathbf{g}_r \otimes \mathbf{g}_z) + \tau_{r\varphi}(\mathbf{g}_z \otimes \mathbf{g}_\varphi + \mathbf{g}_\varphi \otimes \mathbf{g}_z) + 2\tau_{rz}\mathbf{g}_z \otimes \mathbf{g}_z\}.
 \end{aligned} \tag{16}$$

Then, (7) reads

$$\begin{aligned}
 \tau_{rr} &= G, \\
 \tau_{\varphi\varphi} &= G, \\
 \tau_{zz} - 2\lambda\tau_{rz}\frac{\partial v}{\partial r} &= G, \\
 \tau_{r\varphi} &= 0, \\
 \tau_{rz} - \lambda\tau_{rr}\frac{\partial v}{\partial r} &= 0, \\
 \tau_{\varphi z} - \lambda\tau_{r\varphi}\frac{\partial v}{\partial r} &= 0,
 \end{aligned} \tag{17}$$

or, after some rearrangement,

$$\begin{aligned}
 \tau_{rr} &= G, \\
 \tau_{\varphi\varphi} &= G, \\
 \tau_{zz} &= G \left(1 + 2 \left(\lambda \frac{\partial v}{\partial r}\right)^2\right), \\
 \tau_{r\varphi} &= 0, \\
 \tau_{rz} &= \lambda G \frac{\partial v}{\partial r}, \\
 \tau_{\varphi z} &= 0.
 \end{aligned} \tag{18}$$

Substituting these components of the tensor τ into the constitutive law for the components of the Cauchy stress, we obtain

$$\begin{aligned}
 \sigma_{rr} &= -p + G, \\
 \sigma_{\varphi\varphi} &= -p + G, \\
 \sigma_{zz} &= -p + G \left(1 + 2 \left(\lambda \frac{\partial v}{\partial r}\right)^2\right), \\
 \sigma_{r\varphi} &= 0, \\
 \sigma_{rz} &= (\eta + \lambda G) \frac{\partial v}{\partial r}, \\
 \sigma_{\varphi z} &= 0.
 \end{aligned} \tag{19}$$

The obtained stresses can be further substituted in the equations of the linear momentum balance (14), and the latter reduce to

$$\frac{\partial}{r \partial r} \left\{ r(\eta + \lambda G) \frac{\partial v}{\partial r} \right\} = \frac{\partial p}{\partial z}. \quad (20)$$

Adding the boundary conditions

$$v(r = d/2) = 0, \quad \frac{\partial v}{\partial r}(r = 0) = 0, \quad (21)$$

in which d is the internal diameter of the pipe, we complete the formulation of a two-point boundary-value problem.

We note that for $m \rightarrow \infty$ the viscosity is a step function—(5) and (6)—and we can solve the problem analytically as follows:

$$v = \frac{d^2 - 4r^2}{16(\eta + \lambda G)} \frac{\partial p}{\partial z}. \quad (22)$$

This result suggests that the laminar flow of water with a polymer solute is essentially parabolic, similarly to the laminar flow of pure water in a pipe.

Let us analyze this simple solution. First, we assume that there are no polymer molecules in water. Then, we set $\lambda G = 0$ and we get

$$v = \frac{d^2 - 4r^2}{16\eta} \frac{\partial p}{\partial z}. \quad (23)$$

In the linearly viscoelastic mode (5), we obtain the classical Poiseuille flow

$$v = \frac{d^2 - 4r^2}{16\eta_0} \frac{\partial p}{\partial z}. \quad (24)$$

However, when the critical viscous strength is reached with the onset of the ideal fluid mode (6), the velocity becomes uncertain and, generally, tends to a singularity:

$$v \sim \frac{d^2 - 4r^2}{0} \frac{\partial p}{\partial z}. \quad (25)$$

For the critical Reynolds number (~ 2400), an instability happens close to the internal surface of the pipe—we refer the reader to [26] for the details, which are skipped here.

Remarkably, the addition of the polymer solute regularizes the problem even in the case where the viscous strength of water is reached:

$$v = \frac{d^2 - 4r^2}{16\lambda G} \frac{\partial p}{\partial z}. \quad (26)$$

Thus, the addition of the polymer molecules stabilizes the laminar flow and delays the instability which triggers the subsequent transition to turbulence! Such stabilization is provided even by a small amount of the solute. It is worth emphasizing that the addition of large amounts of the solute obviously stabilizes the flow, yet it may also slow down the flow because of the significant increase in viscosity. The latter notion is also in a good qualitative agreement with the experimental data reported by Toms [23].

4 Discussion

Turbulent flow is related to turbulent (non-viscous) friction which, in its turn, slows the motion down and develops high drag. So, it is natural to assume that the drag can be reduced by suppressing turbulent motion and transforming it to the laminar mode. In the latter case, viscous rather than turbulent friction is the main internal process in the flow. Experiments show that such reverse transition from turbulent to laminar flow can be achieved by spreading small amounts of additives in the solvent. Particularly, polymer solutes can help.

In the present note, we showed why an addition of a polymer solute could delay the transition to turbulence or suppress it. For that purpose, we used the modified Navier–Stokes theory with the *viscous strength* for the solvent. The latter theoretical feature allowed us to combine both linearly viscous behavior of the classical Navier–Stokes model characteristic of the laminar flow and the ideal fluid behavior characteristic of the

turbulent flow. Besides, we used the upper-convected Maxwell model for the solute. We solved the problem of the flow in a pipe, and we found that the velocity distribution is parabolic like in the case of the laminar flow of pure Newtonian solvent. However, we found that the material instability related to the loss of viscous friction is suppressed by the polymer solute. Simple analytical formulas show that.

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